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# Application of the Optimal Derivative to the Study of a Ratio-dependent Model describing the Evolution of HIV in Canada

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**Abstract**

In this paper, we propose a study concerning a ratio-dependent model suggested by B. D. Aggarwala which describes the evolution of AIDS in the Canadian society in the case of extinction. On the basis of statistical data on HIV/AIDS published by the authorities of the Canadian health department, B. D. Aggarwala could estimate the number of people infected by this virus in Canada during four years in advance (1996–1999). An application of the optimal derivative as introduced by O. Arino and T. Benouaz enables us to compare the results obtained with those found by B. D. Aggarwala.

**AMS Subject Classifications:** 34A30, 34A34.
**Keywords:** Predator prey, ratio-dependent model, optimal derivative.

## 1 Introduction

The main characteristics of ratio-dependent models used in population models is that the functional responses should depend on the ratio of prey/predator. This type of response corresponds in particular to the sharing of resources among the predators. A

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major problem encountered in this model is the absence of regularity in the neighborhood of the origin. It is not possible to analyze this model using the classical method of linearization. In the study of nonlinear ordinary differential equations, the linearization method plays an important rôle. In [4–7], Arino and Benouaz have introduced an alternative method termed as the optimal derivative method (see also [8–10]). This is an approximation procedure based on the minimization of a certain functional with respect to a curve starting from an initial value $x_0$ and going to zero as $t$ goes to infinity.

Our intention is to apply and make some progress with this procedure in the area of ratio-dependent predator-prey models used in the ecology of population. In the second section, we present the model. The third section is devoted to the review of the optimal derivative. In the last section, we apply the optimal derivative to estimate the number of individuals in the Canadian society infected by the human deficiency virus (HIV) that causes the acquired immune deficiency syndrome (AIDS). For comparison, we also compute the relative errors in our model and in Aggarwala's model (see [1–3]).

## 2 Presentation of the Model

The development of HIV/AIDS in a society can be modelled by a ratio-dependent predatory–prey model. In this model, the population is divided into two classes, namely the HIV negative individuals and the HIV positive ones. Such ratio-dependent systems can be written in the form

$$\begin{cases} \dot{x} = \alpha x(1-x) - \dfrac{xy}{x+y} \\ \dot{y} = -ay + \dfrac{kxy}{x+y}, \end{cases} \tag{2.1}$$

where $x(t)$ is the number of prey (or the HIV negative individuals) at any time $t$, $y(t)$ is the number of predators (or the HIV positive individuals) at any time $t$, $k > 0$ is the conversion factor, $a > 0$ is the death rate of the predator, and $\alpha > 0$ is the growth factor of the prey.

Model (2.1) is used by B. D. Aggarwala for an epidemiologic study concerning the development of HIV/AIDS in Canada [1] (see also [2, 3]). If we simply want to study the behavior of model (2.1) in the origin, we realize that this is not possible because the nonlinear function representing the differential equation is not differentiable in this point; however, in the case of HIV/AIDS, this point represents the annihilation of the society.

On the basis of statistical data on HIV/AIDS which were published by the Canadian health department, B. D. Aggarwala could estimate the number of people infected by this virus in Canada during the years 1996, 1997, 1998, and 1999. A comparison with the actual numbers including the errors made during these years is contained in Table 2.1.

Table 2.1: Estimated and actual numbers of infected individuals during 1996–1999.

| Year | 1996 | 1997 | 1998 | 1999 |
|---|---|---|---|---|
| Estimate | 26130 | 28005 | 30278 | 32306 |
| Actual | 26190 | 28110 | 30181 | 32253 |
| Error $(\%)$ | 0.23 | 0.37 | 0.32 | 0.16 |

In this paper we linearize the model of B. D. Aggarwala by the method of the optimal derivative and thus estimate the number of infected individuals while using the linear model.

# 3 Optimal Derivative Review

Consider a nonlinear ordinary differential problem of the form (see [11–14])

$$\dot{x} = F(x), \quad x(0) = x_0,$$

where

- $x = (x_1, \ldots, x_n)$ is the unknown function,
- $F = (f_1, \ldots, f_n)$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$,

with the assumptions

(H$_1$) $F(0) = 0$,

(H$_2$) the spectrum $\sigma(DF(x))$ is contained in the set $\{z : \mathrm{Re} z < 0\}$ for every $x \neq 0$, in a neighborhood of $0$ for which $DF(x)$ exists,

(H$_3$) $F$ is $\gamma$-Lipschitz continuous.

Given $x_0 \in \mathbb{R}^n$, we choose a first linear map $A_0$. For example, if $F$ is differentiable in $x_0$, then we can take $A_0 = DF(x_0)$ or the derivative value in a point in the vicinity of $x_0$. This is always possible if $F$ is locally Lipschitz. Now, let $y_0$ be the solution of the initial value problem

$$\dot{y} = A_0 y, \quad y(0) = x_0. \tag{3.1}$$

Next, we minimize the functional

$$G(A) = \int_0^\infty \|F(y_0(t)) - A y_0(t)\|^2 \, \mathrm{d}t. \tag{3.2}$$

This minimization problem is always uniquely solvable, and as the optimal linear map minimizing (3.2) we obtain

$$A_1 = \left( \int_0^\infty [F(y_0(t))][y_0(t)]^T \mathrm{d}t \right) \left( \int_0^\infty [y_0(t)][y_0(t)]^T \mathrm{d}t \right)^{-1}.$$

Now we define $y_1$ to be the solution of (3.1) with $A_0$ replaced by $A_1$ and we minimize (3.2) with $y_0$ replaced by $y_1$. Then we continue in this way. The optimal derivative $\widetilde{A}$ is the limit of the sequence build as such (for details, see [4–10]).

# 4 Application

The aim of this application is to estimate the number of people infected by HIV using the optimal derivative. In this application, we use the optimal derivative to study an example which treats the case of extinction. Thus we will confirm the results found previously. We make the same study but we use a model in which we linearize by the method of the optimal derivative. For this we choose an example in the case of extinction.The parameters are given by the model used in [1], i.e., we put the parameters $\alpha = 0.595$, $a = 0.31$, and $k = 0.62$ into the model (2.1) to obtain

$$\begin{cases} \dot{x} = 0.595x(1-x) - \dfrac{xy}{x+y} \\ \dot{y} = -0.31y + \dfrac{0.62xy}{x+y}. \end{cases} \tag{4.1}$$

We choose an initial condition near the origin $(x_0, y_0) = (0.9, 0.399)$ and use the optimal derivative to obtain the matrix

$$\widetilde{A} = \begin{bmatrix} -0.0495 & 0.0628 \\ 0.0061 & -0.0238 \end{bmatrix},$$

which corresponds to the linear system

$$\begin{cases} \dot{x} = -0.0495x + 0.0628y \\ \dot{y} = 0.0061x - 0.0238y. \end{cases} \tag{4.2}$$

In Figures 4.1, 4.2, and 4.3 we compare the two systems (4.1) and (4.2). According to Figures 4.1, 4.2, and 4.3, it is clear that this example treats the case of extinction where $y(t)$ increases at the beginning and then starts to decrease in the course of time. Consequently, $x(t)$ decreases in the course of time until it is extinct. What interests us is the evolution of $y(t)$ since it represents the number of infected people. In particular, we would like to use the optimal derivative to study the variation of $y(t)$ in the interval of reduction.

To use the optimal linear system in order to estimate the number of infected people in Canada, it is initially necessary to calculate the average scale of time which corresponds

Figure 4.1: The phase plan of (4.1) and (4.2) when $(x_0, y_0) = (0.9, 0.399)$.

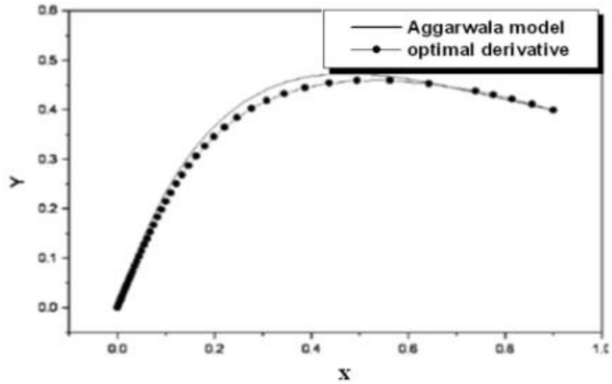



Figure 4.2: The variation of $x$ as a function of time when $(x_0, y_0) = (0.9, 0.399)$.

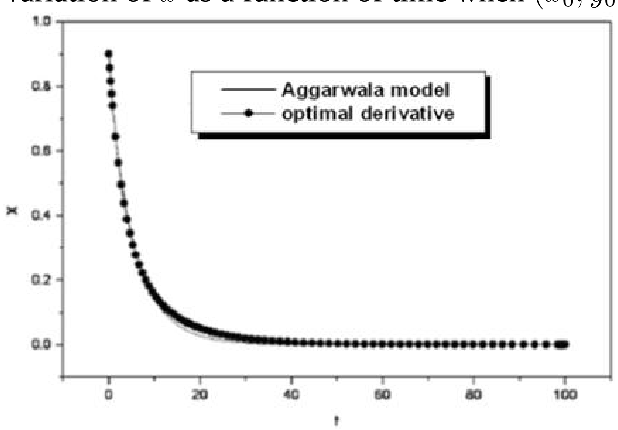
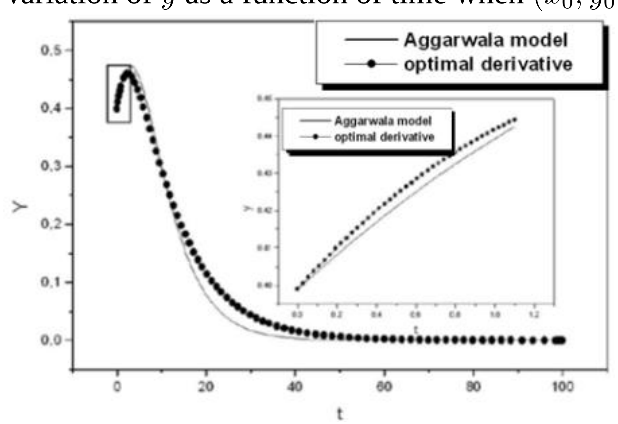



Figure 4.3: The variation of $y$ as a function of time when $(x_0, y_0) = (0.9, 0.399)$.



to one year in Table 2.1. To do this, we observe that the years 1996, 1997, 1998 and even 1999 correspond to the initial conditions of the following year. However, to carry out this study, we consider that the change in population demographics of the Canadian society during these years is constant. In this sense, using the Aggarwala model, we can estimate the value of $t$ that corresponds to the values of $x$ and $y$ of the following year. The values of $x$ and $y$ are provided by the Aggarwala mathematical model. They are those given in Table 4.1. After some calculation, we found that the average time scale which corresponds to one year is equal to $0.23$.

Since $1$ represents the total density of the demographic population of the Canadian society, we define $y_{\mathrm{AM}}$ and $y_{\mathrm{R}}$ by

$$y_{\mathrm{AM}} = \frac{\text{Number of infected individuals using the Aggarwala model}}{30750100}$$

and

$$y_{\mathrm{R}} = \frac{\text{Real (actual) number of infected individuals}}{30750100}.$$

Also,

$$\begin{aligned}\text{Uninfected number} &= \text{Number of total demographic population}\\ &\quad -\text{Infected number by the virus.}\end{aligned}$$

In order to assess how many people are infected with HIV, we use the optimal derivative method. For each year we choose and compute the optimal matrix $\widetilde{A}$ and provide the initial conditions for the following year. Hence each time we take a final time equal to $0.23$. This technique gives remarkable results compared to the other technique, i.e., the case where one considers and uses only one optimal matrix for a single initial condition corresponding to the year 1996. In this case, we must assess the number of infected people each year for the times $0.23$, $0.46$, $0.69$, $0.92$, $\ldots$.

Figure 4.4: Variation of the infected population during the year 1996, $y = f(t)$.

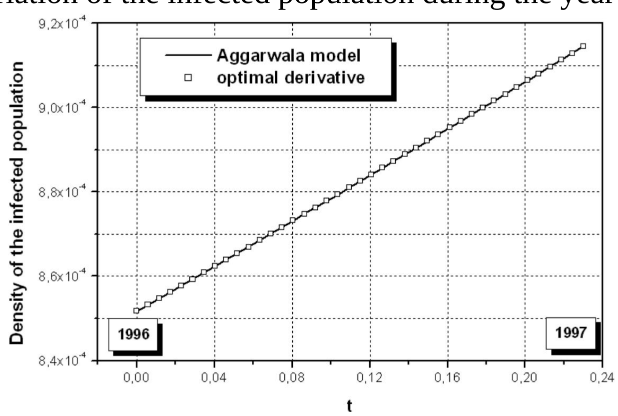


Figure 4.5: Variation of the infected population during the year 1997, $y = f(t)$.

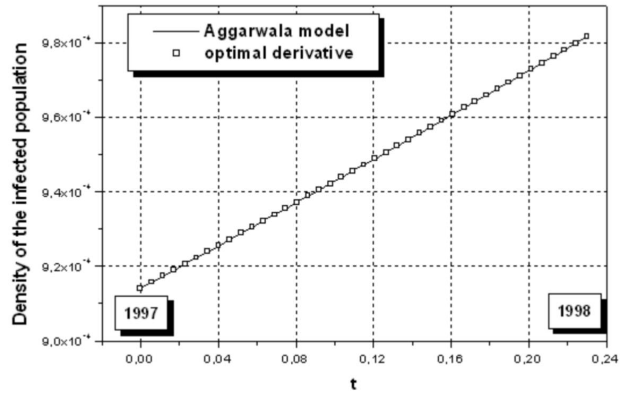
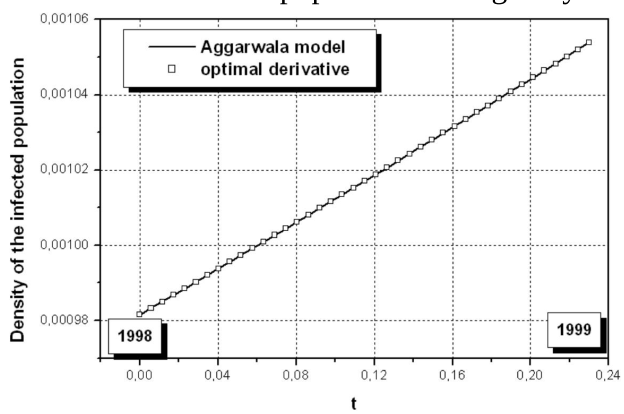



Figure 4.6: Variation of the infected population during the year 1998, $y = f(t)$.




In order to estimate the number of infected individuals by HIV in 1997, we find the optimal matrix as

$$\widetilde{A} = \begin{bmatrix} -0.534089 & -0.998297 \\ 4.497484 \cdot 10^{-7} & 0.308944 \end{bmatrix}.$$

Next, for 1998, the corresponding optimal matrix is

$$\widetilde{A} = \begin{bmatrix} -0.534022 & -0.998172 \\ 5.181080 \cdot 10^{-7} & 0.308867 \end{bmatrix},$$

while for 1999, we find the optimal matrix as

$$\widetilde{A} = \begin{bmatrix} -0.533951 & -0.998038 \\ 5.972634 \cdot 10^{-7} & 0.308783 \end{bmatrix}.$$

For the results of these calculations, we refer to Table 4.1. In Table 4.1, the numbers of infected and uninfected individuals are given using various estimates. These values

Table 4.1: The numbers of infected and uninfected individuals by various estimates.

| Year | 1996 | 1997 | 1998 | 1999 |
|---|---|---|---|---|
| $t$ | 0 | 0.23 | 0.46 | 0.69 |
| $I_{\mathrm{R}}$ | 26190 | 28110 | 30181 | 32253 |
| $U_{\mathrm{R}}$ | 30723910 | 30721990 | 30719919 | 30717847 |
| $y_{\mathrm{R}} \cdot 10^4$ | 8.5170 | 9.1414 | 9.8149 | 10.4887 |
| $x_{\mathrm{R}}$ | 0.999148 | 0.999086 | 0.999019 | 0.998951 |
| $I_{\mathrm{AM}}$ | 26130 | 28005 | 30278 | 32306 |
| $U_{\mathrm{AM}}$ | 30723970 | 30722095 | 30719822 | 30717794 |
| $y_{\mathrm{AM}} \cdot 10^4$ | 8.4975 | 9.1073 | 9.8465 | 10.5060 |
| $x_{\mathrm{AM}}$ | 0.999150 | 0.999089 | 0.999015 | 0.998949 |
| $E$ on $y_{\mathrm{AM}}$ | 0.23 | 0.37 | 0.32 | 0.16 |
| $I_{\mathrm{OD}}$ | 26190 | 28122 | 30183 | 32407 |
| $U_{\mathrm{OD}}$ | 30723910 | 30721990 | 30719919 | 30717847 |
| $y_{\mathrm{OD}} \cdot 10^4$ | 8.5170 | 9.1453 | 9.8156 | 10.5388 |
| $x_{\mathrm{OD}}$ | 0.999148 | 0.999085 | 0.999018 | 0.998946 |
| $E$ on $y_{\mathrm{OD}}$ | 0 | 0.0423 | 0.00685 | 0.479 |
| $E$ on $x_{\mathrm{OD}}$ | 0 | $3.9 \cdot 10^{-5}$ | $7.01 \cdot 10^{-6}$ | $5 \cdot 10^{-4}$ |

Table 4.2: The numbers of infected and uninfected individuals by the optimal derivative.

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|---|
| $t$ | 0 | 0.23 | 0.46 | 0.69 | 0.92 | 1.15 |
| $I_{\mathrm{OD}}$ | 34631 | 37184 | 39925 | 42867 | 46026 | 49417 |
| $U_{\mathrm{OD}}$ | 30715469 | 30712916 | 30710175 | 30707233 | 30704074 | 30700683 |
| $y_{\mathrm{OD}} \cdot 10^4$ | 11.2621 | 12.0924 | 12.9838 | 13.9406 | 14.9678 | 16.0704 |
| $x_{\mathrm{OD}}$ | 0.998874 | 0.998791 | 0.998702 | 0.998606 | 0.998503 | 0.998393 |

are compared with the observed values for the Canadian population. The relative errors are calculated in order to judge the reliability and effectiveness of the optimal derivative method. The relative error is defined by

$$\text{Relative error} = \frac{|\text{Estimated value} - \text{Real value}|}{\text{Real value}}.$$

In order to predict the numbers of infected and uninfected individuals during the years 2000–2005 using the optimal derivative, we use the same technique as described previously. The results are summarized in Table 4.2. In Tables 4.1 and 4.2 we use the following abbreviations: $I$ and $U$ stand for the number of infected individuals and uninfected individuals, respectively. We write AM for the Aggarwala model and OD for the optimal derivative. The observed real data are denoted with R, while $E$ stands for the relative error committed compared to the real data, given in percent.

# 5 Conclusion

Simplification is very important in modelling. The optimal derivative procedure can be used as a powerful tool for modelling predator-prey systems numerically. In this paper we have employed the optimal derivative technique to analyze a ratio-dependent predator-prey model given by B. D. Aggarwala. The optimal derivative method helps to give a quantitative and qualitative description of the behavior of the two populations. We must use the factor time correctly in order to relate theoretical results to the biological reality.

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