LYAPUNOV FUNCTION GENERATED BY LEAST SQUARE APPROXIMATION.*

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Abstract: We apply the least square approximation to a scalar nonlinear o.d.e. This approximation is based on the minimization of a certain functionnal with respect to a curve starting an initial value x_0 and going to 0 as t goes to infinity. It's obtained as a limit of the sequence of a linear maps determined by the procedure. The relation between the obtained approximation and the nonlinear o.d.e was studied. We prove that is a Lyapunov function for the nonlinear equation.

1 INTRODUCTION.

In (Benouaz and Arino) [1] and [2], we have presented a general procedure of approximation of a nonlinear ordinary differential equation. The goal of this talk is to proved that the obtained optimal derivative in the scalar case, is a Lyapunov function for the initial equation [2] and [3].

2 THEORETICAL FRAMEWORK

2.1 FORMULATION OF THE PROBLEM.

Consider the following nonlinear scalar equation

$$\frac{dx}{dt} = f(x(t)), x(0) = x_0, \qquad (1)$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$, and satisfies the following:
$H1)\, f(0) = 0$,
$H2)\, f'(x) < 0$ at every point where $f'(x)$ exists in an interval $]-\alpha, +\alpha[\,, \alpha > 0$.
$H3) f$ is absolutely continuous with respect to the Lebesgue measure.
Our purpose is to apply the optimal derivative, which will associate to system (1) a linear system of the form

$$\frac{dx}{dt} = \tilde{a}x(t), x(0) = x_0 \qquad (2)$$

obtained by minimizing the following functional

$$G(a) = \int_0^{+\infty} |f(x(t)) - ax(t)|\, dt. \qquad (3)$$

For the time being, x is just any function defined on $[0, +\infty[$, bounded, continuous and such that $x \in L^1(0, +\infty)$ and $f(x(\cdot)) \in L^1(0, +\infty)$.
Later on, we will consider functions $x(t)$ that are solution of linear equation.
The minimization of the functional $G(a)$ with respect to a will allow us to get the optimal system (2). Differentiating (3) with respect to a along a function x, yields

$$\frac{\partial G}{\partial a} = 2a \int_0^{+\infty} [x(t)]^2 dt - 2\int_0^{+\infty} [x(t)][f(x(t))]\, dt. \qquad (4)$$

Assuming that a minimizes (3) along a given function x, the above quantities are equal to zero, which leads to

$$a = \frac{\int_0^{+\infty} [x(t)][f(x(t))]\, dt}{\int_0^{+\infty} [x(t)]^2 dt}. \qquad (5)$$

2.2 PROCEDURE.

The formalism presented above will be used iteratively. The initial element of the sequence is the derivative of f at x_0, where x_0 is an arbitrary point in a neighborhood of 0, such that $f'(x_0)$ exists.
Consider system (1)

$$\frac{dx}{dt} = f(x(t)), x(0) = x_0.$$

First step:
Compute $a_0 = f'(x_0)$.
Second step:
Compute a_1 from the solution of the equation

$$\frac{dy}{dt} = a_0 y(t), y(0) = x_0 \qquad (6)$$

*This work was done during the author's visit to the Lab. M.A, C.N.R.S-UA 1204, Université de Pau, 6400 France.

by minimizing the functional

$$G(a) = \int_0^{+\infty} \|f(y(t)) - ay(t)\|^2 dt. \tag{7}$$

y being the solution of eq.(6). a_1 is uniquely determined by formula (5). In order to continue, it is necessary that the above conditions be satisfied at each step.
Let us first assume that this holds. Then the procedure works as follows.

Third step:
Assuming that $a_1, ..., a_{j-1}$ have been computed, to compute a_j from a_{j-1}, we first solve

$$\frac{dy}{dt} = [a_{j-1}]\, y(t), y(0) = x_0. \tag{8}$$

Let y_j be the solution of eq.(8). The minimization of the functional

$$G_j(a) = \int_0^{+\infty} \|f(y_j(t)) - ay_j(t)\|^2 dt \tag{9}$$

yields a_j.
In fact, we have the following relationship between a_{j-1} and a_j.

$$a_j = \frac{\int_0^{+\infty} f(y_j(t))(y_j(t))\, dt}{\int_0^{+\infty} (y_j(t))^2\, dt}. \tag{10}$$

Definition 1 : *If the sequence a_j converges, then the limit $\tilde{a}$ is the optimal derivative of f at x_0.*

3 Analysis of $\tilde{a}(x_0)$.

3.1 Expression.

Now choose $x_0 \in]-\alpha, +\alpha[$ such that $f'(x_0)$ exists. Set $a_0 = f'(x_0)$ and solve the linear equation

$$\frac{dx}{dt} = a_0 x(t), x(0) = x_0, \tag{11}$$

to obtain

$$x(t) = \exp(a_0 t)\, x_0. \tag{12}$$

Replace $x(t)$ in expression (10), we get

$$a_1 = \frac{\int_0^{+\infty} f(e^{a_0 t} x_0)\, e^{a_0 t} dt}{\int_0^{+\infty} e^{2a_0 t} dt} \frac{1}{x_0}. \tag{13}$$

For $x_0 \neq 0$, $f(x(t))$ is almost everywhere differentiable and

$$\frac{d}{dt}[f(e^{a_0 t} x_0)] = f'(e^{a_0 t} x_0)\, e^{a_0 t} x_0 a_0. \tag{14}$$

This gives

$$\int_0^{+\infty} f(x(t))\, e^{a_0 t} dt = \frac{1}{a_0}[f(x(t))\, e^{a_0 t}]_0^{+\infty} - \frac{1}{a_0}\int_0^{+\infty} (f'(x(t))\, e^{2a_0 t} dt)\, x_0 a_0 \tag{15}$$

from which we obtain a_1

$$a_1 = 2\left(\frac{f(x_0)}{x_0} + a_0 \int_0^{+\infty} f'(x(t))\, e^{2a_0 t} dt\right). \tag{16}$$

Changing the variable t to $x(t)$ in the integral, we obtain

$$a_1 = \frac{2}{x_0^2}\int_0^{x_0} f(z)\, dz. \tag{17}$$

So, a_1 does not depend on a_0. Repeating the procedure as indicated above will give the same result. In this case, the procedure leads to the optimal derivative in one step *i.e.*

$$\tilde{a}(x_0) = \frac{2}{x_0^2}\int_0^{x_0} f(z)\, dz. \tag{18}$$

Remark 2 : *Assuming f is analytic*

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \tag{19}$$

we can give another expression of $\tilde{a}(x_0)$

$$\tilde{a}(x_0) = \frac{\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x_0^{n+1} \int_0^{+\infty} e^{(n+1)s a_0} ds}{x_0^2 \int_0^{+\infty} e^{2s a_0} ds}. \tag{20}$$

Finally,

$$\tilde{a}(x_0) = 2\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{(n+1)!} x_0^{n-1}, \tag{21}$$

and

$$\tilde{a}(x_0) = f'(0) + \frac{1}{3} x_0 f''(0) + ... + \frac{2}{(n+1)!} x_0^{n-1} f^{(n)}(0) + ... \tag{22}$$

One can see that the optimal derivative defined by eq.(22) is a sort of mean value of the derivative of f along trajectories linking x_0 to the origin.

3.2 Property.

Lemma 3 : *If the derivative of f exists at 0 and f is continuous, then $\lim_{x_0 \to 0} a^*(x_0) = f'(0)$.*

Proof. With $f(z) = zf'(0) + z\varepsilon(z)$, eq.(18) can be written

$$\tilde{a}(x_0) = f'(0) + \frac{2}{x_0^2}\int_0^{x_0} z\varepsilon(z)\, dz. \tag{23}$$

The second term of eq.(23)

$$\left|\frac{2}{x_0^2}\int_0^{x_0} z\varepsilon(z)\,dz\right| \leq \left|\varepsilon(x_0)\frac{2}{x_0^2}\int_0^{x_0} z dz\right| = |\varepsilon(x_0)| \tag{24}$$

converges to 0 as $x_0 \longrightarrow 0$. Hence $\lim_{x_0\longrightarrow 0}\widetilde{a}(x_0) = f'(0)$. □

We can see that the optimal derivative defined by eq.(23) depends on the initial value x_0 and converges to $f'(0)$ as $x_0 \longrightarrow 0$ if $f'(0)$ exist.

Remark 4 : *It is possible to find a limit even if the derivative of f at 0 does not exist.*

Example 5 : *Consider eq.(18) and write $f(z)$ as follows*

$$f(z) = -zg(z). \tag{25}$$

This yields

$$\widetilde{a}(x_0) = -\frac{2}{x_0^2}\int_0^{x_0} zg(z)\,dz. \tag{26}$$

Let us choose $g(z) = |\sin Log\,|z||$, for $z \neq 0$.

$$\widetilde{a}(x_0) = -\frac{2}{x_0^2}\int_0^{x_0} z\,|\sin Log\,|z||\,dz. \tag{27}$$

Changing z to ux_0,

$$\widetilde{a}(x_0) = -2\int_0^1 u\,|\sin Log\,|ux_0||\,du \tag{28}$$

and changing $-Log(ux_0)$ to v, we have

$$\widetilde{a}(x_0) = -\frac{2}{x_0^2}\int_{\log\frac{1}{x_0}}^{+\infty} e^{-2v}\,|\sin(v)|\,dv \tag{29}$$

and

$$\begin{aligned}\widetilde{a}(x_0) &= -\frac{2}{x_0^2}\int_{k\pi}^{+\infty} e^{-2v}\,|\sin(v)|\,dv \\ &= -\frac{2}{x_0^2}\sum_{l=k}^{\infty}\int_0^{\pi} e^{-2(v+l\pi)}\sin(v)\,dv \\ &= -\frac{2}{x_0^2}\left(\sum_{l=k}^{\infty}e^{-2l\pi}\right)\int_0^{\pi} e^{-2v}\sin(v)\,dv \\ &= -\frac{2}{x_0^2}\frac{e^{-2k\pi}}{1-e^{-2\pi}}\int_0^{\pi} e^{-2v}\sin(v)\,dv.\end{aligned} \tag{30}$$

With $k\pi = Log\frac{1}{x_0} \Longrightarrow e^{-2k\pi} = x_0^2$, we have

$$\widetilde{a}(x_0) = \frac{-2}{1-e^{-2\pi}}\int_0^{\pi} e^{-2v}\sin(v)\,dv. \tag{31}$$

Finally,

$$\widetilde{a}(x_0) = -\frac{2}{5}\coth(\pi). \tag{32}$$

Hence, $\lim_{x_0\longrightarrow 0}\widetilde{a}(x_0)$ exist.

4 Role of $\widetilde{a}(x)$ in the study of the stability.

Proposition 6 *Under the assumptions H1) and H2) in section 2.1 for f, the function*

$$x \longrightarrow v(x) = x^2\,\widetilde{a}(x) \tag{33}$$

is a Lyapunov function for the nonlinear equation.

Proof. Indeed, if $x(t)$ is a solution of eq.(1), differentiating

$$\left[(x(t))^2\,\widetilde{a}(x(t))\right] \tag{34}$$

with respect to t, we obtain

$$\frac{d}{dt}\left[(x(t))^2\,\widetilde{a}(x(t))\right] = (f(x(t)))^2. \tag{35}$$

Since, on the other hand, in view of assumption $H1$ and $H2$ in section 2.1,

$$v(x) < 0. \tag{36}$$

We obtain that $v(x(t)) \longrightarrow 0$ as $t \longrightarrow +\infty$, therefore, $x(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. □

5 Conclusion.

The optimal derivative presented above, allows us to associate a linear equation to a nonlinear o.d.e, even if the derivative of the function f does not exists. The procedure generate a Lyapunov function, and consequently, the role of the optimal derivative in the study of the stability is evidenced.

6 Acknowledgment.

I would like to express my deep thanks to professor O.Arino, who encouraged and supported me during my visit in University of Pau.

References

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