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ON THE RELATIONSHIP BETWEEN THE OPTIMAL DERIVATIVE AND ASYMPTOTIC STABILITY

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Abstract

The aim of this paper is to present the relationship between the optimal derivative and asymptotic stability of a nonlinear ordinary equation. We provide an example where the study of stability of the equilibrium point is a problem; the linearization of the differential equations at the origin has purely imaginary eigenvalues and thus the stability type of the equilibrium point at the origin cannot be deduced from the linear approximation.

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1 Introduction

The study of differential equations is a mathematical field that has historically been the subject of much research, however, continues to remain relevant, by the fact that it is of particular interest in such disciplines as engineering, physical sciences and more recently biology and electronics, in which many models lead to equations of the same type. Most of these equations are generally nonlinear in nature. The term "nonlinear" gathers extremely diverse systems with little in common in their behavior. It follows that there is not, so far,

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a theory of nonlinear equations. A large class of these nonlinear problems is modelled by nonlinear ordinary differential equations.

Routh (1884), Thomson and Tait (1879), and Joukovsky (1882) did not distinguish differential equations from their first-order linear approximations. While acknowledging that the process was not rigorous, and without offering any justification, stability study of the nonlinear system was conducted on the linearized system. A major contribution of Lyapunov and simultaneously, Poincaré, was to provide conditions for the validity of this approximation, based on the properties of the solutions of the linearized model. This method allows us to conclude local results without having to give any quantitative information. In fact, since the advent of the famous memoir of Lyapunov "General Problem of the Stability of Movement" in 1892, numerous studies have been conducted by Hahn (1963), LaSalle and Lefschetz (1961), and many others. However, the determination of a Lyapunov function is a major challenge. Several research methods to find this function have been proposed, for example by Schultz and Gibson (1962), Zoubov (1957), Aizerman and Gantmacher (1962), and others.

The study of stability at the equilibrium point of a nonlinear ordinary differential equation is an almost trivial problem if the function F which defines the nonlinear equation is sufficiently regular in the neighborhood of this point and if its linearization at this point is hyperbolic. In this case, one knows that the nonlinear equation is equivalent to the linearized equation, in the sense that there exists a local diffeomorphism which transforms the neighboring trajectories of the equilibrium point into neighborhoods of zero of the linear equation. On the other hand, the problem is all other when the nonlinear function is nonregular or the equilibrium point is a center. Consider the nonregular case, in particular the case when the only equilibrium point is nonregular. In this case, one cannot derive the nonlinear function and consequently one cannot study the linearized equation. A natural question arises then: Is it possible to associate another linear equation to the nonlinear equation which has the same asymptotic behavior?

2 Idea of the Problem

The idea proposed by Benouaz and Arino is based on the method of approximation. In [2–6] they introduced the optimal derivative, which is in fact a global approximation in contrast to the nonlinear perturbation of a linear equation, having distinguished behavior with respect to the classical linear approximation in the neighborhood of the stationary point. The approach used is the least square approximation.

We examine, in what follows, the relationship between the concept of Lyapunov function, or more generally the stability of an equilibrium point, and the properties of the optimal derivative at this point. To illustrate further the relationship between the optimal derivative and the nonlinear equation, we show an example in which the Jacobian of the linearization associated with a nonlinear equation has only eigenvalues with strictly negative real part in the vicinity of 0 (except at $x = 0$), and the solution does not tend to 0. There is no theoretical result that could in such a case determine the nature of the stability of the origin, and the example illustrates that the behavior depends on the equation. The procedure of the optimal derivative can detect those equations for which there is stability.

The aim of this paper is to present some results about asymptotic stability obtained via the optimal derivative. After a brief review of the optimal derivative procedure in Section 3, Section 4 is devoted to the study of the relationship between the optimal derivative (in comparison with the classical linearization) and asymptotic stability. As an application, we consider two examples in Section 5, while Section 6 presents a result about the sign of the trace of a relevant matrix. A bifurcation analysis is offered in Section 7 and some conclusions are given in Section 8.

3 Review of the Optimal Derivative

Consider a nonlinear ordinary differential problem of the form

$$\dot{x} = F(x), \quad x(0) = x_0,$$

where

- $x = (x_1, \ldots, x_n)$ is the unknown function,
- $F = (f_1, \ldots, f_n)$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$,

with the assumptions

(H$_1$) $F(0) = 0$,

(H$_2$) the spectrum $\sigma(DF(x))$ is contained in the set $\{z : \operatorname{Re} z < 0\}$ for every $x \neq 0$, in a neighborhood of 0 for which $DF(x)$ exists,

(H$_3$) F is γ-Lipschitz continuous.

Given $x_0 \in \mathbb{R}^n$, we choose a first linear map A_0. For example, if F is differentiable in x_0, then we can take $A_0 = DF(x_0)$ or the derivative value in a point in the vicinity of x_0. This is always possible if F is locally Lipschitz. Now, let y_0 be the solution of the initial value problem

$$\dot{y} = A_0 y, \quad y(0) = x_0. \tag{3.1}$$

Next, we minimize the functional

$$G(A) = \int_0^\infty \|F(y_0(t)) - Ay_0(t)\|^2 \, \mathrm{d}t. \tag{3.2}$$

This minimization problem is always uniquely solvable, and as the optimal linear map minimizing (3.2) we obtain

$$A_1 = \left(\int_0^\infty [F(y_0(t))][y_0(t)]^T \mathrm{d}t \right) \left(\int_0^\infty [y_0(t)][y_0(t)]^T \mathrm{d}t \right)^{-1}. \tag{3.3}$$

Now we define y_1 to be the solution of (3.1) with A_0 replaced by A_1 and we minimize (3.2) with y_0 replaced by y_1. Then we continue in this way. The optimal derivative $\widetilde{A}$ is the limit of the sequence build as such (for details see [2–5]).

4 Optimal Derivative and Asymptotic Stability

Although the stability criteria by linearization are clearly stated and rigorously justified, classical linearization is sometimes inconvenient because it assumes that the Jacobian matrix at the equilibrium point exists. However, this assumption is not always true. Consider for instance a nonlinear system with a function involving an absolute value such that the nonlinearity is not differentiable in the vicinity of the equilibrium point. The classical linearization gives a necessary condition but not a sufficient one, since it does not allow to study stability in the presence of purely imaginary eigenvalues. The search for a Lyapunov function itself constitutes a sensitive issue because it is based in general on experience and luck.

4.1 Scalar case

Remark 4.1. We have proved in [5] that the optimal derivative

$$\tilde{a}(x_0) = \frac{2}{x_0^2}\int_0^{x_0} f(x)dx$$

permits to construct a Lyapunov function of the form

$$v(x) = x^2\tilde{a}(x).$$

The scalar case is very interesting in the sense that we can write the optimal derivative as a function of the classical linearization of f at 0 (if f' exists at 0).

4.2 Vectorial case

In the vectorial case, and for better illustration of the connection between the optimal derivative and the nonlinear equation, one can consider an example characterizing a class of nonlinear, nondifferentiable functions in which the Jacobian of the associated linearization has eigenvalues with strictly negative real part in the vicinity of 0, except at 0, and the solutions do not tend towards 0. There is no theoretical result that allows to conclude from this assumption the nature of stability of the origin. The example illustrates the fact that the result depends on the equation. The procedure of optimal derivative makes it possible to detect those equations for which there is stability.

Example 4.2. Consider

$$\dot{x}_1 = -x_1 g(x_1), \quad \dot{x}_2 = -|x_1|^n x_2 + |x_1|^{n-1}, \tag{4.1}$$

where $n \geq 2$ and $g(x_1) > 0$. If $x_2 \to 0$, then $\dot{x}_2 > 0$, and if in addition $x_2(0) > 0$, then $x_2(t) > 0$. The Jacobian of the linearization associated with equation (4.1) can be written for $n = 2$ and $g(x_1) = \frac{x_1^2}{2}$ as

$$DF(x_1,x_2) = \begin{bmatrix} -3x_1^2/2 & 0 \\ a(x_1,x_2) & -|x_1|^2 \end{bmatrix}$$

(with appropriate a), a matrix whose determinant is positive and whose trace is negative. Therefore the real parts of all eigenvalues of the Jacobian associated with (4.1) are strictly negative, and thus (4.1) is asymptotically stable except at the origin. With $x_0 = (1,0)$, the calculation of the optimal derivative gives at the first step

$$A_1 = \begin{bmatrix} -0.340938 & 0.227282 \\ 0.909119 & -0.170407 \cdot 10^{-4} \end{bmatrix},$$

whose eigenvalues are

$$\lambda_1 = -0.65595 \quad \text{and} \quad \lambda_2 = 0.314995.$$

The optimal derivative applied to this example gives at the first step a matrix with one positive and one negative eigenvalue. The result obtained by using the optimal derivative is thus in conformity with the result observed.

5 Application

When the eigenvalues of the linearization at an equilibrium point are purely imaginary, the local dynamics about the equilibrium point cannot be determined by the linear approximation. Indeed, depending on the nonlinear terms, the equilibrium can be unstable, stable, or even asymptotically stable. Consequently, we will investigate the effects of the nonlinear terms in equations of the form

$$\dot{x} = F(x) = Mx + G(x), \quad x(0) = x_0.$$

We consider the case when the nonlinear terms are given by

$$G(x) = a\Phi(\|x\|)x, \quad x = (x_1, x_2) \quad \text{with} \quad \Phi(\|x\|) = \mathrm{o}(\|x\|), \quad a = \pm 1.$$

Therefore the function F defining the nonlinear differential equation can be written as

$$F(x) = Mx + a\,[\mathrm{o}(\|x\|)]\,x,$$

and thus an example may be given as [8]

$$\dot{x}_1 = x_2 + ax_1\left(x_1^2 + x_2^2\right), \quad \dot{x}_2 = -x_1 + ax_2\left(x_1^2 + x_2^2\right). \tag{5.1}$$

In this case, the study of stability of the equilibrium point is difficult as the linearization of the system at the origin has purely imaginary eigenvalues and thus the stability type of the equilibrium point at the origin cannot be deduced from the linear approximation. Depending on the value of the constant a, we consider the following two examples.

Example 5.1 (System (5.1) with $a = -1$). System (5.1) can be rewritten as

$$\dot{x}_1 = x_2 - x_1\left(x_1^2 + x_2^2\right), \quad \dot{x}_2 = -x_1 - x_2\left(x_1^2 + x_2^2\right). \tag{5.2}$$

The classical linearization of F at the equilibrium point $(0,0)$ gives

$$DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5.3}$$

with eigenvalues $\lambda_{1,2} = \pm i$. The classical linearization shows that the equilibrium point in the origin is stable but not asymptotically stable and it is a center. Thus it cannot give information about the detailed geometry of the trajectories.

With

$$DF(0,0.1) = \begin{bmatrix} -10^{-2} & 1 \\ -1 & -3\cdot 10^{-2} \end{bmatrix},$$

the computational procedure gives (with $\varepsilon = 10^{-6}$)

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} -5.0822\cdot 10^{-3} & 1 \\ -1 & -5.0942\cdot 10^{-3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -5.0822\cdot 10^{-3} & 0 \\ 0 & -5.0942\cdot 10^{-3} \end{bmatrix}, \end{aligned} \tag{5.4}$$

which has eigenvalues

$$\lambda_1^* = -5.0882\cdot 10^{-3} + i \quad \text{and} \quad \lambda_2^* = -5.0882\cdot 10^{-3} - i. \tag{5.5}$$

The real parts of both λ_1^* and λ_2^* are negative. Thus the optimal linearization is asymptotically stable and shows the origin as a focus. Therefore the origin is asymptotically stable.

To check the nonlinear equation, we take as a Lyapunov function the quadratic, positive definite, function

$$V(x) = x_1^2 + x_2^2 \quad \text{for} \quad x = (x_1, x_2).$$

The computation of the derivative of V along a trajectory gives

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2.$$

By replacing $\dot{x}_1$ and $\dot{x}_2$ by the equations in (5.2), we get

$$\dot{V}(x) = -2\left(x_1^2 + x_2^2\right)^2 < 0.$$

Thus $\dot{V}(x)$ is negative definite, and, in addition, $V(x)$ is decreasing (more precisely, $\|x\| \to 0$ implies $V(x) \to 0$) and unbounded in absolute value (i.e., $\|x\| \to \infty$ implies $V(x) \to \infty$). Thus the equilibrium at the origin is globally asymptotically stable.

Figures 1, 2, and 3 show, respectively, the vector fields of the classical linear system (5.3), the linear system (5.4), and the nonlinear system (5.2). Clearly, the classical linearization presents a center. The vector fields of the nonlinear system and the optimal linear system are identical and lead to the same conclusion, showing the origin as a focus, hence being asymptotically stable.

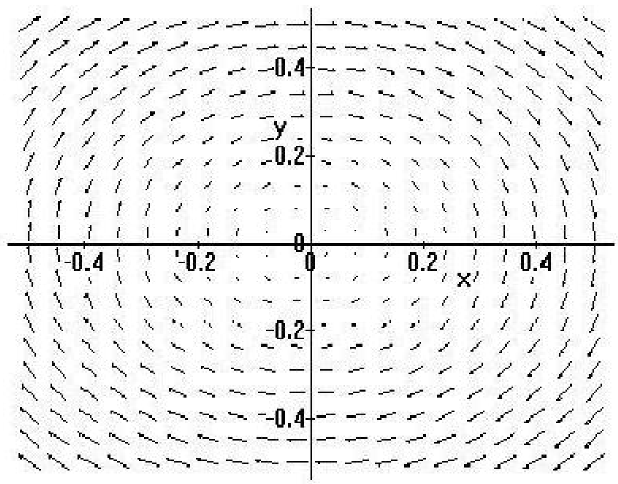
Example 5.2 (System (5.1) **with** $a = 1$**).** System (5.1) can be rewritten as

$$\dot{x}_1 = x_2 + x_1\left(x_1^2 + x_2^2\right), \quad \dot{x}_2 = -x_1 + x_2\left(x_1^2 + x_2^2\right). \tag{5.6}$$

In this case, the integral

$$G(A) = \int_0^\infty \|F(x(t)) - Ax(t)\|^2\,\mathrm{d}t$$

Figure 1. Vector field of the classical linearization (5.3)



is not convergent. We therefore integrate from 0 to $-\infty$, i.e.,

$$G(A) = \int_0^{-\infty} \|F(x(t)) - Ax(t)\|^2 \, \mathrm{d}t.$$

The optimal derivative procedure gives (with $\varepsilon = 10^{-6}$)

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} 5.08221 \cdot 10^{-3} & 1 \\ -1 & 5.094196 \cdot 10^{-3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 5.0822 \cdot 10^{-3} & 0 \\ 0 & 5.09416 \cdot 10^{-3} \end{bmatrix}, \end{aligned} \tag{5.7}$$

which has the eigenvalues

$$\lambda_1^* = 5.088205 \cdot 10^{-3} + i \quad \text{and} \quad \lambda_2^* = 5.088205 \cdot 10^{-3} - i. \tag{5.8}$$

The real parts of both λ_1^* and λ_2^* are positive. Thus the optimal linearization is unstable and shows the origin as a focus. Therefore the origin is asymptotically instable.

Figures 4 and 5 show, respectively, the vector fields of the optimal linear system (5.7) and the nonlinear system (5.6).

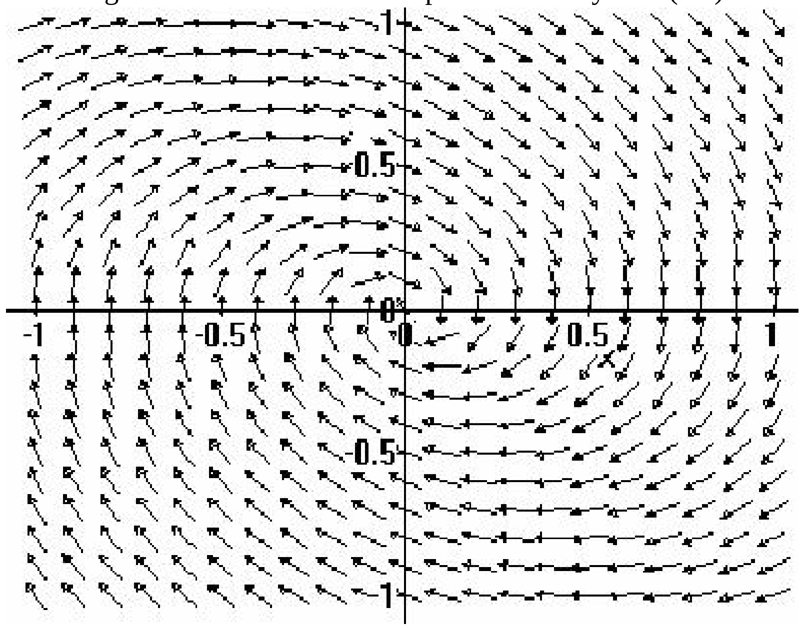
Remark 5.3. In the two examples that we have just seen, it should be noted that the nonlinear system is written after application of the optimal linearization as

$$\widetilde{A} = M + r(x_0), \quad x_0 = x(0), \tag{5.9}$$

where $M = DF(0)$ and

$$r(x_0) = \left[\int_0^{\infty} G\left(e^{tA}x_0\right) \left[e^{tA}x_0\right]^T \mathrm{d}t \right] \left[\int_0^{\infty} \left[e^{tA}x_0\right] \left[e^{tA}x_0\right]^T \mathrm{d}t \right]^{-1}.$$

Figure 2. Vector field of the optimal linear system (5.4)



The first term is the linearization. The second term, which is actually the optimal linearization of the nonlinear function G, turns out to be dependent on the initial value x_0. It is as if we had perturbed $DF(0)$, writing the optimal matrix in the form

$$\widetilde{A} = DF(0) + \mathrm{o}(\|x_0\|).$$

6 Perturbation of DF(0)

We consider

$$\dot{x} = F(x) = Mx + G(x), \quad x(0) = x_0,$$

where

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$x \in \mathbb{R}^2$, and $G(x) = a\Phi(\|x\|)x$ with $\Phi(z) > 0$ whenever $z > 0$ such that $\Phi(\|x\|) = \mathrm{o}(\|x\|)$.

Let $x_0 \neq 0$. On the set of two-by-two matrices with real entries we define the map φ by

$$\varphi(A) = \left[\int_0^\infty F\left(e^{tA}x_0\right)\left[e^{tA}x_0\right]^T \mathrm{d}t\right][\Gamma(A)]^{-1} = M + r(A),$$

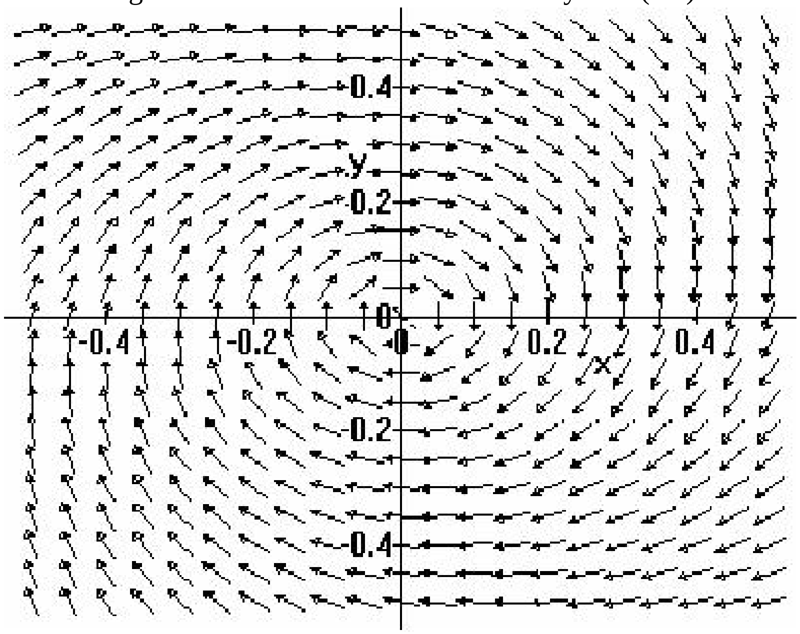
where

$$\Gamma(A) = \int_0^\infty \left[e^{tA}x_0\right]\left[e^{tA}x_0\right]^T \mathrm{d}t$$

and

$$r(A) = \left[\int_0^\infty G\left(e^{tA}x_0\right)\left[e^{tA}x_0\right]^T \mathrm{d}t\right][\Gamma(A)]^{-1}.$$

Figure 3. Vector field of the nonlinear system (5.2)



Now we build $\widetilde{A} = \varphi(A_0)$, i.e.,

$$\widetilde{A} = \varphi(A_0) = M + r(A_0),$$

the initial matrix A_0 being $A_0 = DF(x_0)$. It is assumed that G was chosen such that the spectrum of A_0 is close to the spectrum of M. We choose x_0 close to 0 so that $r(A_0)$ is very small in a neighborhood of x_0. So $e^{tA_0}x_0$ tends to 0 exponentially as $t \to \infty$, and $\Gamma(A_0)$ is positive definite.

Theorem 6.1. *Under the assumptions of this section, if G is as small as necessary and DG is uniformly bounded, then*

$$\operatorname{sgn}\left(\operatorname{tr}(\widetilde{A})\right) = \operatorname{sgn}(a).$$

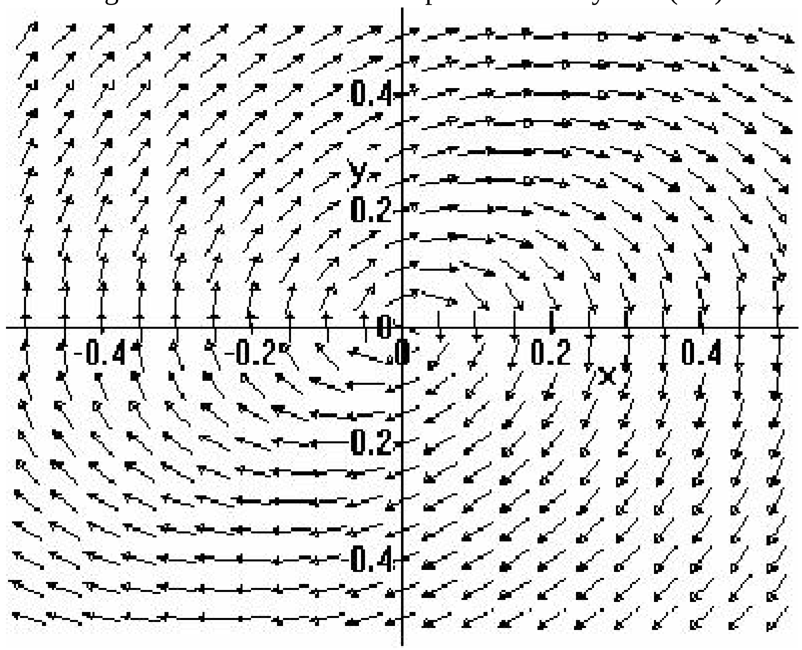
Proof. The calculation of the trace of $\widetilde{A}$ gives

$$\begin{aligned}
\operatorname{tr}(\widetilde{A}) &= \operatorname{tr}\left(M + \left[\int_0^\infty G\left(e^{tA_0}x_0\right)\left(e^{tA_0}x_0\right)^T \mathrm{d}t\right][\Gamma(A_0)]^{-1}\right) \\
&= \operatorname{tr}(M) + \operatorname{tr}\left(\left[\int_0^\infty G\left(e^{tA_0}x_0\right)\left(e^{tA_0}x_0\right)^T \mathrm{d}t\right][\Gamma(A_0)]^{-1}\right) \\
&= \operatorname{tr}\left(\left[\int_0^\infty G\left(e^{tA_0}x_0\right)\left(e^{tA_0}x_0\right)^T \mathrm{d}t\right][\Gamma(A_0)]^{-1}\right)
\end{aligned}$$

as $\operatorname{tr}(M) = 0$. Using the property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we obtain

$$\begin{aligned}
\operatorname{tr}(\widetilde{A}) &= \int_0^\infty \operatorname{tr}\left([\Gamma(A_0)]^{-1}\left[G\left(e^{tA_0}x_0\right)\left(e^{tA_0}x_0\right)^T\right]\right)\mathrm{d}t \\
&= \int_0^\infty \operatorname{tr}\left(\left([\Gamma(A_0)]^{-1} G\left(e^{tA_0}x_0\right)\right)\left(e^{tA_0}x_0\right)^T\right)\mathrm{d}t.
\end{aligned}$$

Figure 4. Vector field of the optimal linear system (5.7)



Now since $[\Gamma(A_0)]^{-1} G\left(e^{tA_0}x_0\right) =: C$ is a column vector and $\left(e^{tA_0}x_0\right)^T =: L$ is a row vector, and by taking account of $\mathrm{tr}(CL) = \mathrm{tr}(LC) = LC$, we find

$$\mathrm{tr}(\widetilde{A}) = a\int_0^\infty \Phi\left(\left\|e^{tA_0}x_0\right\|\right)\left(e^{tA_0}x_0\right)^T [\Gamma(A_0)]^{-1}\left(e^{tA_0}x_0\right)\mathrm{d}t.$$

The terms inside this last integral are positive, and thus $\mathrm{tr}(\widetilde{A})$ depends on the sign of a so that the sign of the trace is related to that of the perturbation G by

$$\mathrm{sgn}\left(\mathrm{tr}(\widetilde{A})\right) = \mathrm{sgn}(a).$$

This completes the proof. □

Remark 6.2. Moreover, by a calculation already realized in the work of Benouaz and Arino [1], we have

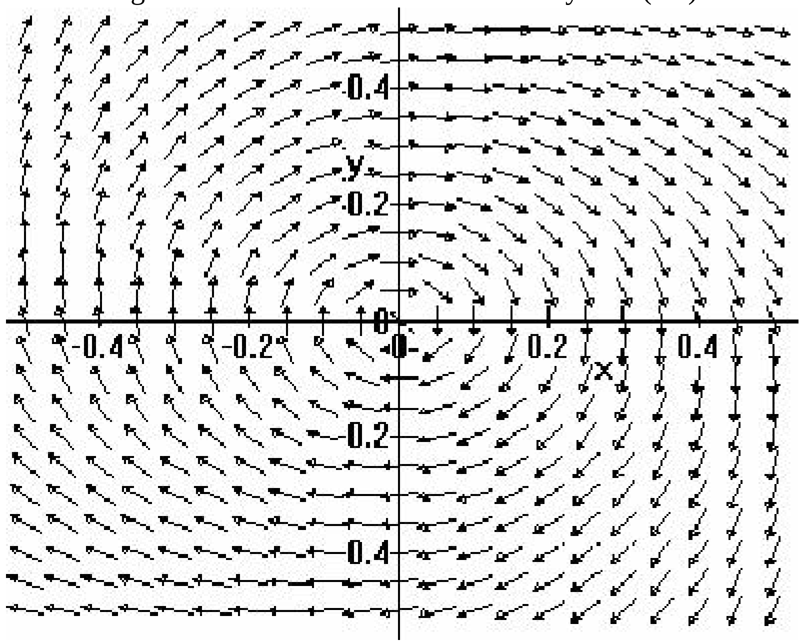
$$\mathrm{tr}(r(\widetilde{A})) \le aC\left(\mathrm{O}(\|x_0\|)\right) \le a\mathrm{O}(\|x_0\|).$$

This shows the influence of the initial conditions on the study of the stability of the equilibrium point under consideration. This result is important in the sense that it suggests the possibility to find a result for stability (in the case of center) by studying the trace of the optimal matrix. Further examples show similar results.

7 Bifurcation Analysis

In this section we use the numerical continuation and bifurcation package AUTO2000 [7] in order to analyze the stability and the branches of periodic solutions in the neighborhood

Figure 5. Vector field of the nonlinear system (5.6)



of the nonhyperbolic equilibrium point zero with purely imaginary eigenvalues of the nonlinear system (5.1), where a is taken as bifurcation parameter. The presented numerical results show the existence of an Andronov–Hopf bifurcation for the value of the parameter $a = 0$.

In order to compare the results of the optimal derivative method to those obtained from the analysis by the numerical continuation and bifurcation package AUTO2000, we analyze the nonlinear system by changing the variables to polar coordinates. Let

$$x_1 = r\cos(\theta) \quad \text{and} \quad x_2 = r\sin(\theta).$$

To derive a differential equation for r, we note

$$x_1^2 + x_2^2 = r^2 \quad \text{so that} \quad x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}.$$

Using the equations (5.1), we obtain

$$\begin{aligned} r\dot{r} &= x_1\left(x_2 + ax_1(x_1^2+x_2^2)\right) + x_2\left(-x_1 + ax_2(x_1^2+x_2^2)\right) \\ &= a\left(x_1^2+x_2^2\right)^2 = ar^4. \end{aligned}$$

so that

$$\dot{r} = ar^3.$$

Using this and again the first equation in (5.1), we find

$$\begin{aligned} x_2 + ax_1r^2 &= x_2 + ax_1\left(x_1^2+x_2^2\right) = \dot{x}_1 = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) \\ &= ar^3\cos(\theta) - r\dot{\theta}\sin(\theta) = ax_1r^2 - \dot{\theta}x_2 \end{aligned}$$

Table 1. First Run

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)
1	1	EP	1	-1.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1	10	HB	2	1.00000E-17	0.00000E+00	0.00000E+00	0.00000E+00
1	20	EP	3	1.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00

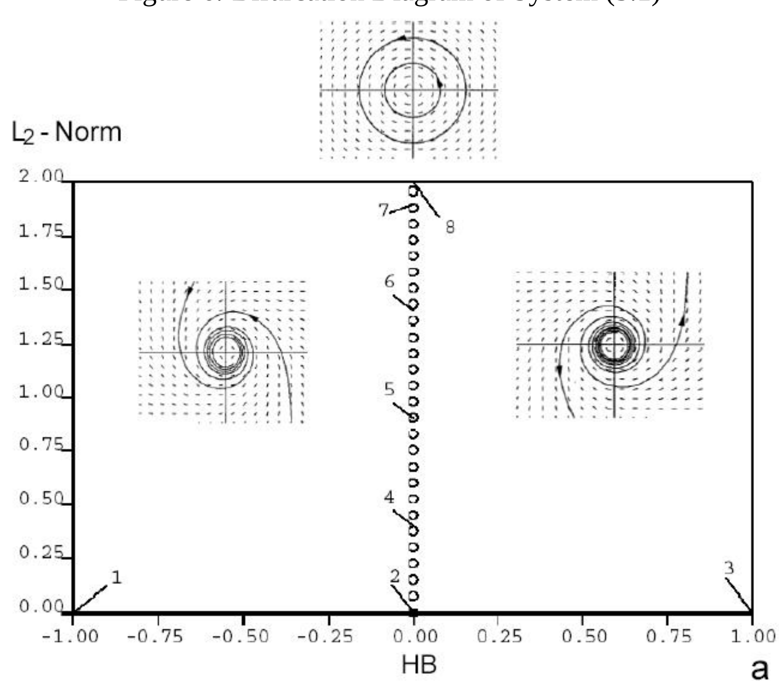
so that

$$\dot{\theta} = -1.$$

Thus the radial and angular motions are independent. This shows that if $a < 0$, then $r(t) \to 0$ monotonically as $t \to \infty$. In this case, the origin is an asymptotically stable spiral. If $a = 0$, then $r(t) = r_0$ for all t and the origin is a center. Finally, if $a > 0$, then $r(t) \to \infty$ monotonically and the origin is an unstable spiral.

Using the numerical continuation and bifurcation package AUTO2000, we obtain Figures 6 and 7, representing the bifurcation diagram for system (5.1) with a as the bifurcation parameter. Executing Auto2000 gives the information in Table 1 (first run) and Table 2 (second run). In Tables 1 and 2, BR stands for branching point, PT for the point number, TY for the solution type, PAR(0) for the bifurcation parameter a, LAB for the label of the solutions, and U(1), U(2) for the variables x_1 and x_2. The bifurcation diagram Figure 6

Figure 6. Bifurcation Diagram of System (5.1)



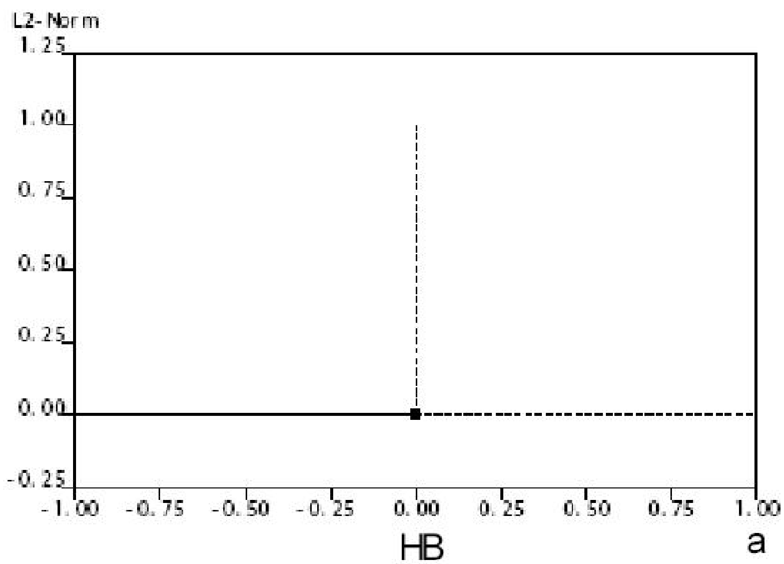
shows that when $a < 0$, all solutions spiral clockwise into the origin with increasing t. In this case the origin is an asymptotically stable spiral (see Figure 2 obtained by the opti-

Table 2. Second Run

BR	LAB	PAR(0)	L2-NORM	MAX(1)	MAX(2)	PERIOD
-2	4	1.149038E-14	4.00000E-01	3.999710E-01	3.999660E-01	6.283185
-2	5	2.480256E-15	9.00000E-01	8.999145E-01	8.998845E-01	6.283185
-2	6	9.969503E-16	1.40000E+00	1.399865E+00	1.399828E+00	6.283185
-2	7	5.206941E-16	1.90000E+00	1.899815E+00	1.899772E+00	6.283185
-2	8	4.38281E-16	2.00000E+00	1.99981E+00	1.99976E+00	6.283195

mal derivative system). For $a = 0$, at the point **HB**, the character of the solutions changes. **HB** stands for Hopf bifurcation (Table 1) with change of stability. This change is usually accompanied with the appearance of a periodic orbit encircling the equilibrium point. In this case, all solutions are periodic so that the origin is a center. Since at this value of the parameter there are periodic orbits encircling the origin, these solutions are unstable or their stability is unknown (labeled respectively 4, 5, 6, 7, 8 in Table 2). For $a > 0$ the origin becomes unstable and all solutions spiral clockwise without bounds (see Figure 5). Figure 7 is the summary of the stability analysis obtained by Auto2000 (more detailed re-

Figure 7. Bifurcation diagram. Solid curves: stable solution. Dashed curves: unstable solutions and solutions of unknown stability



sults are saved in the numerical data-files) and a confirmation of the results obtained using the optimal derivative of the two-dimensional system (5.1), because for $a < 0$ the origin is asymptotically stable (the solid curve in Figure 7) and the real parts of the eigenvalues (5.5) are negative, and for $a > 0$ the real parts of the eigenvalues (5.8) become positive and the origin is unstable (dashed curve in Figure 7).

8 Conclusion

In conclusion, the answer to the question relative to the relation between the property of stability of the linear equation obtained by the optimal derivative and that of the nonlinear equation in the vectorial case is very subtle. Generally, when the procedure converges, the matrix obtained is stable. All these considerations bring us to the following conjecture.

Conjecture 8.1. If the procedure of the optimal derivative converges and the limit of the sequence A_j is exponentially stable (or if A_j has a stable fixed point), then the nonlinear system is stable.

This study shows that the conditions under which the conjecture was formulated can be satisfied, i.e., the existence, uniqueness and convergence towards a stable fixed point [4]. The procedure of calculation also enables us to solve problems where the classical linearization may not be useful. The bifurcation analysis using the numerical continuation and bifurcation package AUTO2000 confirms the results obtained by the optimal derivative. It also shows its potential to be a tool for analyzing the stability of this type of two-dimensional nonlinear ordinary differential equations.

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