Computational Approach of the Optimal Linearization of the Nonlinear O.D.E: Application to Nonlinear Electronic Circuit

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Abstract— **The aim of this paper is to present a generalization of the optimal linearization. This method enables us to associate a linear map to a nonlinear ordinary differential equation. Our results show clearly the existence and the unicity of the best optimal linearization in the sense of the least square. We use an approach to associate a linear optimal equation to a nonlinear equation in the neighbourhood of zero, even though the equation cannot be linearized around the origin using the Frechet derivative. An application is done to analyze the behaviour of an electronic circuit..**

***Index Terms*— Electronic circuit, Least square approximation, Nonlinear ODE, Optimal linearization.**

I. INTRODUCTION

The linearization method plays an important role in the analysis of systems (ex: Electronic circuit in engineering...), modeled by nonlinear ordinary differential equation. The principal method when studying behavior and stability of solutions of an ordinary differential equation in the neighborhood of an equilibrium point considers the linear equation obtained by differentiating (at the Frechet sense) the nonlinearity of a nonlinear equation at this point. A similar behavior is encountered in the hyperbolic case.
However, there are three setbacks to this method [2], [1].
1. If the nonlinearity is not smooth enough in the neighborhood of a stationary point, then, in general, one cannot compute the Frechet derivative.
2. The derivative of the nonlinear equation to be equal zero.

3. Case where the eigenvalues are purely imaginary.

The behavior of the solution of the nonlinear equation in the neighbourhood of such a point can be anything. In the present work, we propose a method which associates a linear map to a nonlinear ordinary deferential equation near the equilibrium point, defined as a generalization of the optimal linearization. It is a sort of a global linearization by opposition to the nonlinear perturbation of a linear equation, which is different from the classical linearization in the vicinity of a stationary point. The following approach is the type of optimization. Our results are in the line of the work by Vujanovic [12] and Jordan et al.[7], [8]. The first sections are devoted to present a general formalism to apply the optimal linearization method in the scalar and vectorial forms. In the last sections, an application is considered to illustrate the theoretical procedure with comments.

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II. THEORETICAL FRAMEWORK

A. Formulation of the problem

Consider the following nonlinear ordinary differential equation

$$\frac{dx}{dt} = F(x(t)), x(0) = x_0 \tag{1}$$

Where: $x = (x1, ..., xn)$ is the unknown function.
$F = (f1, ..., fn)$ is a given function on an open subset of $IR^{n,}$ with the assumptions:

H1) $F(0) = 0$.
H2) The spectrum $\sigma(DF(x))$ is contained in the set $\{z : \operatorname{Re} z < 0\}$ for every $x = 0$, in a neighbourhood of 0, for which DF (x) exists.
H3) F is γ Lipschitz continuous.
Our purpose can be formulated as follows:
Find a linear ordinary differential equation of the form

$$\frac{dx}{dt} = \tilde{A}\,x(t)\,,\; x(0) = x_0. \tag{2}$$

Where $\tilde{A} \in M_n(IR)$, is to be determined in such a way that it has the same behavior that the nonlinear equation (1), both (1) and (2) having the same initial value, by minimizing with respect to A the functional

$$G(A) = \int_0^{+\infty} \| F(x(t)) - A\,x(t) \|^2\, dt \tag{3}$$

$A \in M_n(IR)$.

At the beginning, x is just any function defined on $[0,+\infty[$, bounded, continuous such that $x \in L^1(0,+\infty)$ and $F(x(\cdot)) \in L^1(0,+\infty)$. Later on, we will consider the

function $x(t)$ that is solution of a linear equation.
This approach is the optimization in the least square sense.
The existence and unicity of the solution $\tilde{A}$ in the least square sense are guaranteed by general theorems of approximation [11], [10] and [6].

B. *Formalism*

Differentiating the functional (3) with respect to A along a function x, yields

$$DG(A)\alpha = 2\int_0^{+\infty} \langle Ax(t) - F(x(t)), \alpha\, x(t)\rangle\, dt \quad (4)$$

for every matrix α. In particular, for matrices α such as $\alpha_{l,m} = 1$; $\alpha_{ij} = 0$, if $(i,j) \neq (l,m)$.

We have

$$\int_0^{+\infty} \langle Ax(t) - F(x(t)), \alpha\, x(t)\rangle\, dt = \\ = \int_0^{+\infty} [Ax(t) - F(x(t))]_l \;\; x_m(t)\, dt \quad (5)$$

Assuming that A minimizes (3) along a given function x, the quantities

$$\int_0^{+\infty} [Ax(t) - F(x(t))]_l \;\; x_m(t)\, dt, \forall 1 \le l, m \le n \quad (6)$$

are equal to zero, which leads to

$$\sum_{j=1}^{n} a_{l,j} \left(\int_0^{+\infty} x_j(t) x_m(t)\, dt\right)_{1\le j,m\le n} = \left(\int_0^{+\infty} f_l(x(t))\, x_m(t)\, dt\right)_{1\le l,m\le n} \quad (7)$$

with obvious notations for the elements of matrix A.

Introducing valued function $\Gamma(x)$ defined by

$$\Gamma(x) = \int_0^{+\infty} [x(t)][x(t)]^T\, dt = \left(\int_0^{+\infty} x_j(t) x_m(t)\, dt\right)_{1\le j,m\le n} \quad (8)$$

and assuming that Γ (x) is non singular, we obtain

$$A = \left[\int_0^{+\infty} [F(x(t))][x(t)]^T\, dt\right][\Gamma(x)]^{-1} \quad (9)$$

If the inverse matrix of Γ exists, then A is uniquely determined.

C. *Procedure*

The resolution procedure is implemented in two steps. The initial matrix is the Jacobian matrix of F at x_0, where x_0 is an arbitrary point in a neighborhood of 0, such that $DF(x_0)$ exists.

Consider the system (1)

$$\frac{dx}{dt} = F(x(t)),\; x(0) = x_0$$

First step:

Compute $A_0 = DF(x_0)$

Second step:

To compute $\tilde{A}$ from A_0, we first solve

$$\frac{dx}{dt} = A_0\; y(t),\; y(0) = x_0 \quad (10)$$

letting y_0 to be the solution of (10). The minimization of the functional in the least square sense

$$G(A) = \int_0^{+\infty} \| F(y(t)) - A\, y(t)\|^2\, dt. \quad (11)$$

yields $\tilde{A}$.

$\tilde{A}$ is uniquely determined by formula (9). It can be written as

$$\tilde{A} = \left[\int_0^{+\infty} [F(e^{A_0 t}x_0)][e^{A_0 t}x_0]^T dt\right] \times \left[\int_0^{+\infty} [e^{A_0 t}x_0][e^{A_0 t}x_0]^T dt\right]^{-1} \quad (12)$$

By definition, $\tilde{A}$ provides the optimal linearization of F at x_0.

III. Properties of the Method

A. *Case where the application F is linear*

If F is linear with σ (F) in the negative part of the complex plane, (7) reads

$$A\Gamma(x) = F\Gamma(x) \quad (13)$$

It is clear that $A = F$ is solution. It is unique if the inverse of $\Gamma(x)$ exists.

The optimal linearization of linear system is the system itself.

B. *General case*

Consider the more general system of nonlinear equations with a nonlinearity of the form

$$F(x) = Mx + \tilde{F}(x), x(0) = x_0 \quad (14)$$

where M is linear.

The computation of the matrix $\tilde{A}$ gives

$$\tilde{A} = \left[\int_0^{+\infty} [F(x(t))][x(t)]^T\, dt\right][\Gamma(x)]^{-1}, \quad (15)$$

which can be written as

$$\tilde{A} = \left[M\Gamma(x) + \left(\int_0^{+\infty} [\tilde{F}(x(t))][x(t)]^T\, dt\right)\right][\Gamma(x)]^{-1}, \quad (16)$$

And finally

$$\tilde{A} = M + \left[\int_0^{+\infty} [\tilde{F}(x(t))][x(t)]^T\, dt\right][\Gamma(x)]^{-1}, \quad (17)$$

Hence, $\tilde{A} = M + \tilde{A}_1$ with

$$\tilde{A}_1 = \left[\int_0^{+\infty} [\tilde{F}(x(t))][x(t)]^T\, dt\right][\Gamma(x)]^{-1}, \quad (18)$$

If, in particular, some components of F are linear, then the corresponding components of $\tilde{F}$ are zero

And those of $\widetilde{A}$ and F are same. If f_k is linear, then the k^{th} row of the matrix $\widetilde{A}$ is equal to f_k.

Scalar case

We will now give the expression of the optimal linearization in the scalar case. For this, consider the following nonlinear scalar equation

$$\frac{dx}{dt} = f(x(t)),\ x(0) = x_0 \qquad (19)$$

Where $f : IR \to IR$ and satisfies the following conditions

$H1)$ $f(0) = 0$.

$H2)$ $f'(0) < 0$ at every point where $f'(x)$ exists in an interval $\left]-\alpha, +\alpha\right[, \alpha > 0$.

$H3)$ f is absolutely continuous with respect to the Lebesgue measure.

Choose $x_0 \in \left]-\alpha, +\alpha\right[$ such that $f'(x_0)$ exists.

Set $a_0 = f'(x_0)$ and use the method presented in section 2.3.

We solve the linear equation

$$\frac{dx}{dt} = a_0 x(t),\ x(0) = x_0 \qquad (20)$$

Whose solution is

$$x(t) = e^{(a_0 t)} x_0 \qquad (21)$$

Substituting f for F in expression (9), we get

$$\widetilde{a} = \frac{\left(\int_0^{+\infty} f\left(e^{a_0 t} x_0\right) e^{a_0 t} dt\right)}{\int_0^{+\infty} e^{a_0 t} dt} \frac{1}{x_0} \qquad (22)$$

For $x_0 \neq 0$, $f(x(t))$ is almost everywhere differentiable and

$$\frac{d}{dt}\left[f\left(e^{a_0 t} x_0\right)\right] = f'\left(e^{a_0 t} x_0\right) e^{a_0 t} x_0 a_0, \qquad (23)$$

This gives

$$\int_0^{+\infty} f(x(t)) e^{a_0 t} dt = \frac{1}{a_0}\left[f(x(t)) e^{a_0 t}\right]_0^{+\infty} - \frac{1}{a_0}\int_0^{+\infty}\left(f'(x(t)) e^{2a_0 t} dt\right) x_0 a_0, \qquad (24)$$

From which we obtain $\widetilde{a}$

$$\widetilde{a} = 2\left(\frac{f(x_0)}{x_0} + a_0 \int_0^{+\infty} f'(x(t)) e^{2a_0 t} dt\right). \qquad (25)$$

Changing the variable t to $x(t)$ in the integral, we obtain

$$\widetilde{a}(x_0) = \frac{2}{x_0}\int_0^{x_0} f(z) dz. \qquad (26)$$

$\widetilde{a}(x_0)$ is the result of the optimal linearization in the scalar case.

Remark: The role of the optimal approximation in the study of stability [5] is evidenced in the scalar case by the fact that in this case, the function $x \to v(x) = x^2 \widetilde{a}(x)$ is a Lyapunov function for the nonlinear equation (19).Indeed, if $x(t)$ is a solution of equation (19), differentiating $\left[(x(t))^2 \widetilde{a}(x(t))\right]$ With respect to t, we obtain

$$\frac{d}{dt}\left[(x(t))^2 \widetilde{a}(x(t))\right] = \left(f(x(t))\right)^2 \qquad (27)$$

Since, on the other hand, $v(x) < 0$ (in view assumption $H1$ and $H2$ in section 4), we obtain that $v(x(t)) \to 0$ as $t \to +\infty$, therefore, $x(t) \to 0$ as $t \to +\infty$.

IV. Order of linearization

To estimate the order of the linearization, we will evaluate the functional

$$G(A) = \int_0^{+\infty} \left\| F(y(t)) - A y(t) \right\|^2 dt \qquad (28)$$

Where A is any matrix. Starting from an arbitrary matrix A_0, the optimal matrix giving the optimal linearization is obtained by minimizing the functional

$$G(A) = \int_0^{+\infty} \left\| F(y_0(t)) - A y_0(t) \right\|^2 dt. \qquad (29)$$

Where $y_0(t)$ is the solution of equation

$$\frac{dy}{dt} = A_0 y(t)$$

We have the following relation ship between $\widetilde{A}$ and A

$$\int_0^{+\infty} \left\| F(\widetilde{y}(t)) - \widetilde{A}\widetilde{y}(t) \right\|^2 dt \leq \leq \int_0^{+\infty} \left\| F(\widetilde{y}(t)) - A\widetilde{y}(t) \right\|^2 dt$$

where $\widetilde{y}(t)$ is the solution of equation

$$\frac{dy}{dt} = \widetilde{A} y(t).$$

Consequently

$$\int_0^{+\infty} \left\| F(\widetilde{y}(t)) - \widetilde{A}\widetilde{y}(t) \right\|^2 dt = \inf_{\forall A \in M(R)} \int_0^{+\infty} \left\| F(\widetilde{y}(t)) - A\widetilde{y}(t) \right\|^2 dt$$

$\forall A \in M(R)$, such as $RE\sigma(A) \subset \left]-\infty, 0\right[$.

In particular, for $A = DF(0)$, we have

$$\int_0^{+\infty} \left\| F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t) \right\|^2 dt \le$$
$$\le \int_0^{+\infty} \| F(\tilde{y}(t)) - DF(0)\tilde{y}(t) \|^2 \, dt$$

With the assumptions

$$\|\tilde{y}(t)\| \le C\|x_0\| \text{ and } \| F(x) - DF(0)x \| = O\left(\|x\|^2\right),$$

we obtain

$$\int_0^{+\infty} \left\| F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t) \right\|^2 dt \le O\left(\|x_0\|^2\right)^2 \quad (30)$$

We will now evaluate the difference $\|x(t) - y(t)\|$ where x is the solution of eq. (1) and $\tilde{y}$ the solution of the optimal linear equation, both having the same initial value. We have

$$\frac{dx}{dt} - \frac{dy}{dt} = F(x(t)) - \tilde{A}\tilde{y}(t) \qquad (31)$$
$$= F(x(t)) - F(\tilde{y}(t)) + F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t).$$

From assumption $H3$ in section 4, we have

$$\frac{d}{dt}\|x(t) - \tilde{y}(t)\| \le \gamma\|x(t) - \tilde{y}(t)\| + \left\|F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t)\right\|, \qquad (32)$$

and using the Gronwall's lemma, we obtain

$$\|x(t) - \tilde{y}(t)\| \le \int_0^t e^{\gamma t}\left\|F(\tilde{y}(s)) - \tilde{A}\tilde{y}(s)\right\| ds$$
$$\le \left(\int_0^t e^{2\gamma(t-s)} ds\right)^{\frac{1}{2}} \left(\int_0^t \left\|F(\tilde{y}(s)) - \tilde{A}\tilde{y}(s)\right\|^2 ds\right)^{\frac{1}{2}}. \qquad (33)$$

For every $T > 0$, there exists $M \ge 0$ such that:

$$\|x(t) - \tilde{y}(t)\| \le M\|x_0\|^2 \text{ for } 0 \le t \le T \quad (34)$$

And every x_0 in the neighbourhood of 0, independent of T.

The optimal linearization method is of order two or higher with respect the initial value.

More generally, it has the same order as the nonlinearity.

V. Application

In this section, we present an example to illustrate the usefulness of the theory presented in section (2). This is the case where the nonlinearity is not smooth enough near the steady state and consequently, the system cannot be linearized at 0 using the classical linearization.

Consider the circuit in Fig.1.

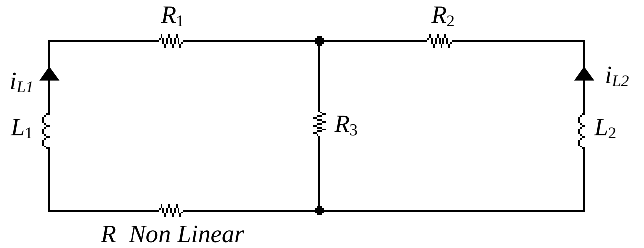

Fig.1. Electronic circuit with a nonlinear resistance

This circuit contains a nonlinear resistance whose characteristic (Fig.2.) is represented by a non regular function (absolute value type)

$$v_N = R_0\left(i_N + |i_N|\right) \qquad (35)$$

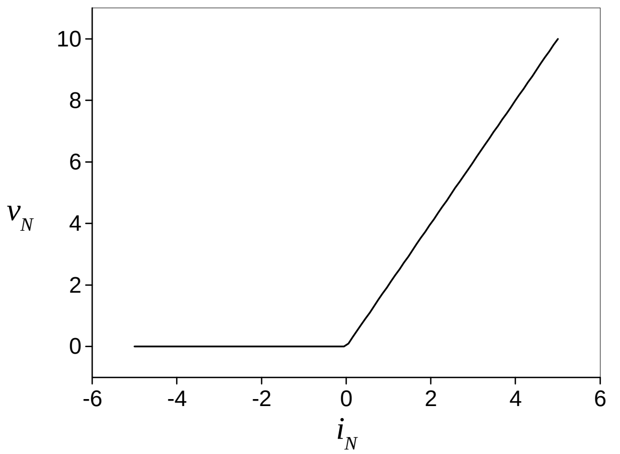

Fig.2. Nonlinear characteristic of the resistance

The equations of state defining this system are

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{R_1 + R_3 + R_0}{L_1}x - \dfrac{R_0}{L_1}|x| - \dfrac{R_3}{L_1}y \\[2ex] \dfrac{dy}{dt} = -\dfrac{R_3}{L_2}x - \dfrac{R_2 + R_3}{L_2}\,y, \end{cases} \qquad (36)$$

Where

$$x = i_{L_1}$$
$$y = i_{L_2}, \qquad (37)$$

We normalize the component values of the circuit to 1 and the system (36) becomes

$$\begin{cases} \dfrac{dx}{dt} = -3x - |x| - y \\[2ex] \dfrac{dy}{dt} = -x - 2y. \end{cases} \qquad (38)$$

We compute first the Jacobian matrix $DF(x)$.

For $(x_0, y_0) = (0.5, 0.3)$

$$DF\left(x_0 \,, y_0\right) = \begin{bmatrix} -4 & -1 \\ -1 & -2 \end{bmatrix}, \tag{39}$$

the optimal derivative can be written

$$\tilde{A} = \begin{bmatrix} -3.9524 & -1.0686 \\ -1 & -2 \end{bmatrix}. \tag{40}$$

Illustrate graphically the results obtained above, the Fig.4, Fig.5 and Fig.6, represent, as function of time, respectively, the solutions $\left(x(t), y(t)\right)$ of the nonlinear system (38), compared to the solutions of the optimal linear system (40), for the initial conditions $\left(x_0 , y_0\right) = \left(0.5, 0.3\right)$. And the quadratic error between the nonlinear system (38) and the optimal linear system (40).

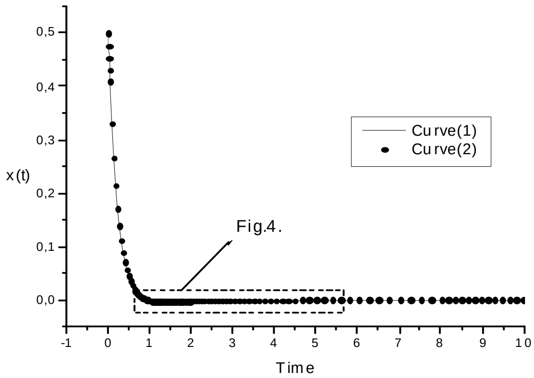

Fig.4. The variation of the solution x(t) as a function of time for the initial conditions $\left(x_0 , y_0\right) = \left(0.5, 0.3\right)$ Curve (1): corresponds to the solution of system (38)

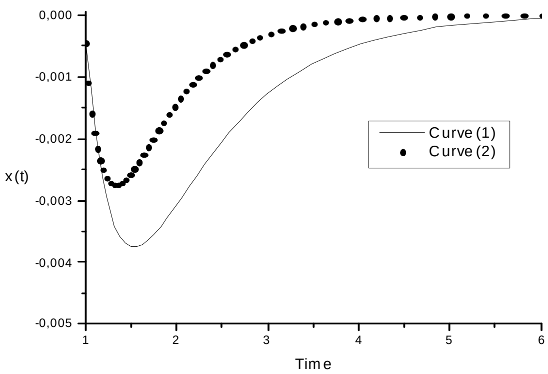

Fig.5. The Zoom of a part of the solutions x(t) Shown in the Fig 4.

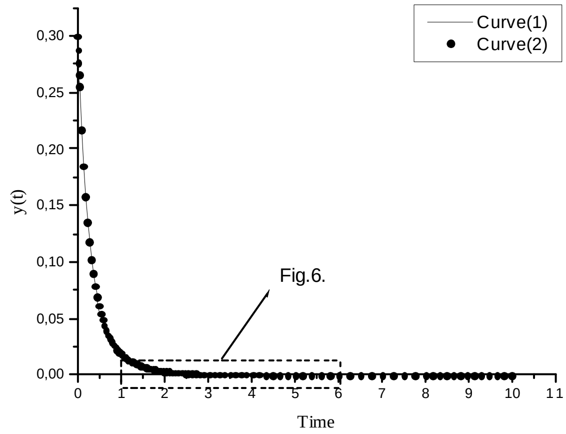

Fig.6. The variation of the solution y (t) as a function of time for the initial conditions $\left(x_0 , y_0\right) = \left(0.5, 0.3\right)$
Curve (1): corresponds to the solution of system (38)
Curve (2): corresponds to the solution of system (40)

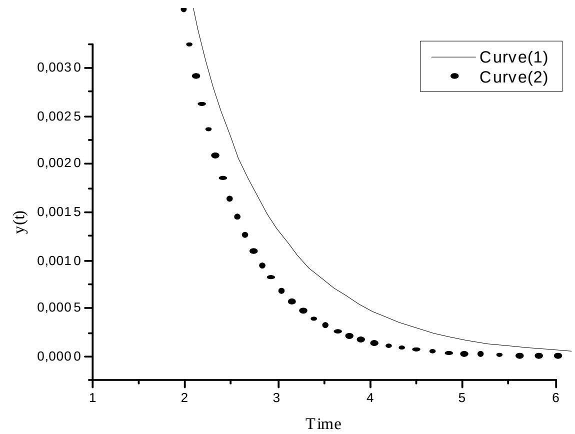

Fig.7. The Zoom of a part of the solutions y(t) Shown in the Fig.6.

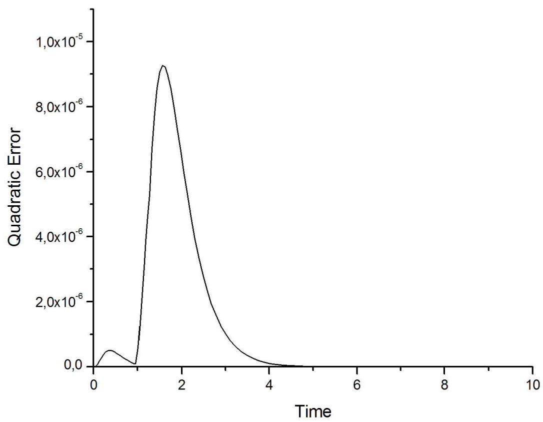

Fig.8. the quadratic error as function of time between the nonlinear system (38) and the optimal linear system (40)

VI. Discussions

The example presented in section 6 shows the opportunity given by the optimal linearization to study behavior of the nonlinear system in the neighborhood of 0 when the classical linearization can be anything. In fact, the Fig.4, Fig. 6, (solutions), Fig.8. (quadratic error) presented satisfactory adequacy of approximate results compared to the exact ones. This is confirmed by the computation of the quadratic error which never exceeds 0.09 %, even if there is a variation between the optimal linear and nonlinear solutions represented in Fig.5., Fig.7. Corresponding respectively to the zoom of part of the solutions $x(t)$, and $y(t)$. Our gaol is to explain trough this both figures, the increase in the curve representing the quadratic error (Fig.8.) in the time interval [1]-[6], which remain in the order of 10^{-5}, and that this variation have not an influence on the approximation quality, and its effectiveness to approach the nonlinear system (38), by a adequate optimal linear system (40).

VII. Conclusion

We have presented in this paper further development regarding the generalization of the optimal linearization method. Our main results confirm the existence and the unicity of the best optimal linearization in the sense of the least square. The order of the linearization method is two or higher with respects the initial value, and is generally of the same order as the nonlinearity. The method enables us to associate a linear optimal equation to a nonlinear equation in the neighbourhood of 0, even though the latter equation cannot be linearized around the origin using the Frechet derivative. This is the case notably when the functions involved are not smooth near the origin.

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The Magneto-Optic Rotation in Magnetised Plasma Study of Magneto-Optic Isolator" Mediterranean Winter, ICTON-MW'08, 2nd ICTON, 11-13 Dec. 2008, pp.1-5, IEEE, ISBN : 978-1-4244-8.

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