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Least-Square Approximation of a Nonlinear O.D.E. with Excitation

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Abstract—The aim of this paper is to present a computational procedure of an optimal approximation method for a nonlinear ordinary differential equation with excitation based on the minimization in the least-square sense. The approximation is of order two or higher with respect to the initial value. We provide an application which contained an example with two kinds of excitations: continuous and periodic. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Least square approximation, Computational procedure, Excitation, Quadratic error, Continuous, Periodic.

1. INTRODUCTION

In [1–5] Benouaz and Arino have presented a computational procedure which yields a linear map defined as the optimal approximation of a nonlinear ordinary differential equation of the following form:

$$\frac{dx(t)}{dt} = F(x(t)), \qquad x(0) = x_0.$$

They have applied this method to some problems when the classical linearization cannot be used (behavior and stability of solutions). In particular, they have applied this procedure to a specific nonlinear ordinary differential equation for which they proved existence, uniqueness, and convergence of the optimal approximation associated with it.

The work presented in [2,6–8] is based on the applicability of the proposed method to the study of the stability.

In [9], they gave a necessary and sufficient condition for uniqueness of the elements of the sequence determined in the course of the optimal approximation and proved that the order is two or higher.

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In this paper, we propose to extend this procedure to a nonlinear ordinary differential equation with excitation. This class encompasses various systems such as electronic circuits, mechanical systems, aerodynamical systems, and population dynamics systems. Our purpose is to study a system governed by a set of equations which can be written in the form [10]

$$\frac{dx}{dt} = F(x(t), u(t)), \qquad x(0) = x_0.$$

We consider especially the decoupled (x, u) case. The excitation is denoted by $u(t)$ [11] (the external force which can be supplied for example by a photovoltaic generator). Our results are in the line of previous work by Vujanovic [12,13] and Jordan *et al.* [14–16].

We give a brief overview of the contents. The next two sections are devoted to preliminaries and a general formalism with properties of the optimal procedure. In Section 4, the order of the approximation in the case of constant excitation is studied.

Finally, we illustrate the applicability of the procedure through an example. The results are discussed using the calculus of the quadratic error.

2. THEORETICAL FRAMEWORK

2.1. Position of the Problem

Consider the following system of nonlinear ordinary differential equations:

$$\frac{dx}{dt} = F(x(t), u(t)), \qquad x(0) = x_0, \tag{1}$$

where $x = (x_1, \dots, x_n)$ is the unknown function, $u = (u_1, \dots, u_n)$ is the external excitation which can be constant (continuous) or a function of time (periodic), and $F = (f_1, \dots, f_n)$ is a given function on an open subset Ω of $\mathbb{R}^n$. Our purpose is to find a linear ordinary differential equation of the form

$$\frac{dx}{dt} = \tilde{A}x + \tilde{B}u, \qquad x(0) = x_0, \tag{2}$$

which has the same behavior as the nonlinear differential equation (1), both (equations (1) and (2)) have the same initial value. $(\tilde{A}, \tilde{B}) \in \mathcal{M}_n(\mathbb{R})$ are to be determined. For this, we shall assume the following.

(H1) $F(0, 0) = 0$.

(H2) The spectrum $\sigma(DF(x))$ is contained in the set $\{z : \operatorname{Re} z < 0\}$ for every $x \neq 0$, in a neighborhood of 0, for which $DF(x)$ exists.

(H3) F is γ Lipschitz continuous with respect to x.

System (2) will give an optimal approximation to system (1), starting from the initial value x_0 and going to the steady state as t goes to infinity.

2.2. Formalism

Consider the functional defined by

$$G(A, B) = \int_0^{+\infty} \|F(x(t), u(t)) - Ax(t) - Bu(t)\|^2 \, dt, \tag{3}$$

where F is defined on an open subset Ω of $\mathbb{R}^n$, $(A, B) \in \mathcal{M}_n(\mathbb{R})$ are to be determined successively. Here, x is just any function defined on $[0, +\infty[$, bounded, continuous, and such that $(x, u) \in L^1(0, +\infty)$ and $F(x(\cdot), u(\cdot)) \in L^1(0, +\infty)$. Later on, we will consider a function $x(t)$ that is the solution of a linear equation. This approach is the optimization in the least-square sense. The existence and unity of the solution (A, B) in the least-square sense are guaranteed by the general theorems of approximation [17–21].

Differentiating $G(A, B)$ with respect to A along a function x, and with respect to B along a function u, yields

$$\begin{aligned} DG(A)\alpha &= 2\int_0^{+\infty} \langle Ax(t) + Bu(t) - F(x(t), u(t)), \alpha x(t)\rangle \, dt, \\ DG(B)\beta &= 2\int_0^{+\infty} \langle Ax(t) + Bu(t) - F(x(t), u(t)), \beta u(t)\rangle \, dt, \end{aligned} \tag{4}$$

for every matrix α and β. In particular, for matrices such that

$$\alpha_{l,m} = 1; \quad \alpha_{ij} = 0, \quad \beta_{l,m} = 1; \quad \beta_{ij} = 0, \qquad \text{if } (i, j) \neq (l, m), \tag{5}$$

we have

$$\begin{aligned} &\int_0^{+\infty} \langle Ax(t) + Bu(t) - F(x(t), u(t)), \alpha x(t)\rangle \, dt \\ &\qquad = \int_0^{+\infty} [Ax(t) + Bu(t) - F(x(t), u(t))]_l x_m(t) \, dt, \\ &\int_0^{+\infty} \langle Ax(t) + Bu(t) - F(x(t), u(t)), \beta u(t)\rangle \, dt \\ &\qquad = \int_0^{+\infty} [Ax(t) + Bu(t) - F(x(t), u(t))]_l u_m(t) \, dt. \end{aligned} \tag{6}$$

First, assuming that A minimizes (3) along a given function x, the above quantities are equal to zero, i.e.,

$$\left(\int_0^{+\infty} [Ax(t) + Bu(t) - F(x(t), u(t))]_l x_m(t) \, dt\right)_{\forall\, 1 \le l,m \le n} = 0. \tag{7}$$

Let (a_{ij}) denote the elements of matrix A and (b_{ij}) those of matrix B, then (6) yields

$$\begin{aligned} \sum_{j=1}^{n} \left[a_{l,j} \left(\int_0^{+\infty} x_j(t) x_m(t) \, dt\right)_{1 \le j,m \le n} + b_{l,j} \left(\int_0^{+\infty} u_j(t) x_m(t) \, dt\right)_{1 \le j,m \le n} \right] \\ = \left(\int_0^{+\infty} f_l(x(t), u(t)) x_m(t) \, dt\right)_{1 \le j,m \le n}. \end{aligned} \tag{8}$$

Let

$$\begin{aligned} \Gamma_A(x) &= \int_0^{+\infty} [x(t)][x(t)]^\top \, dt = \left(\int_0^{+\infty} x_j(t) x_m(t) \, dt\right)_{1 \le j,m \le n}, \\ \Phi_A(u, x) &= \int_0^{+\infty} [u(t)][x(t)]^\top \, dt = \left(\int_0^{+\infty} u_j(t) x_m(t) \, dt\right)_{1 \le j,m \le n}, \\ \Psi_A(x, u) &= \int_0^{+\infty} [F(x(t), u(t))][x(t)]^\top \, dt, \end{aligned} \tag{9}$$

then,

$$A\Gamma_A(x) + B\Phi_A(u, x) = \Psi_A(x, u). \tag{10}$$

Assuming that B minimizes (3) along a given function u, the system of equations (6) is equal to zero, i.e.,

$$\left(\int_0^{+\infty} [Ax(t) + Bu(t) - F(x(t), u(t))]_l u_m(t) \, dt\right)_{\forall\, 1 \le l,m \le n} = 0. \tag{11}$$

The system of equations (6) yields

$$\sum_{j=1}^{n}\left[a_{l,j}\left(\int_0^{+\infty} x_j(t)u_m(t)\,dt\right)_{1\le j,m\le n} + b_{l,j}\left(\int_0^{+\infty} u_j(t)u_m(t)\,dt\right)_{1\le j,m\le n}\right] = \left(\int_0^{+\infty} f_l(x(t),u(t))u_m(t)\,dt\right)_{1\le j,m\le n}. \tag{12}$$

Denoting

$$\begin{aligned}\Gamma_B(u) &= \int_0^{+\infty}[u(t)][u(t)]^\top dt = \left(\int_0^{+\infty} u_j(t)u_m(t)\,dt\right)_{1\le j,m\le n},\\ \Phi_B(x,u) &= \int_0^{+\infty}[x(t)][u(t)]^\top dt = \left(\int_0^{+\infty} x_j(t)u_m(t)\,dt\right)_{1\le j,m\le n},\\ \Psi_B(x,u) &= \int_0^{+\infty}[F(x(t),u(t))][u(t)]^\top dt,\end{aligned} \tag{13}$$

we obtain

$$A\Phi_B(x,u) + B\Gamma_B(u) = \Psi_B(x,u). \tag{14}$$

Equations (10) and (14) allow us to write the matrices A and B as follows:

$$\begin{aligned}A &= [\Psi_A(x,u) - (B)\Phi_A(u,x)][\Gamma_A(x)]^{-1},\\ B &= [\Psi_B(x,u) - (A)\Phi_B(x,u)][\Gamma_B(u)]^{-1}.\end{aligned} \tag{15}$$

We have implicitly assumed that matrices Γ_A and Γ_B are nonsingular and consequently A and B are uniquely defined if $\Gamma_A(x)$ and $\Gamma_B(u)$ are invertible.

2.3. Procedure

DIAGRAM OF THE LEAST-SQUARE APPROXIMATION PROCEDURE. The diagram of the least-square approximation procedure is given in Figure 1.

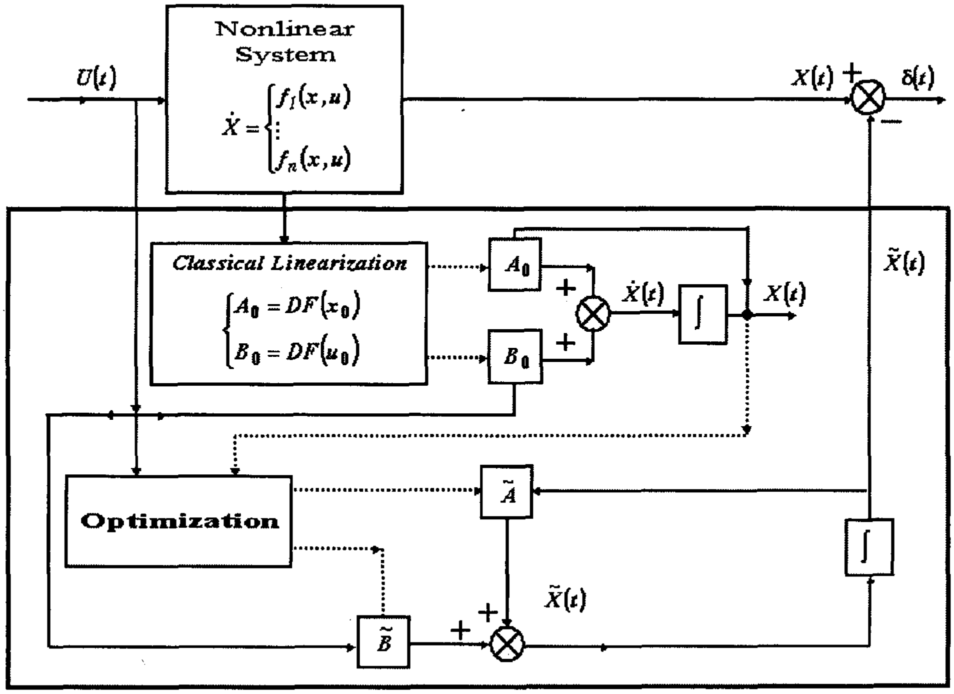

Figure 1. Diagram of the least-square approximation procedure.

ALGORITHM. The computation presented above will be used iteratively. We shall assume that the successive matrices A_j and B_j are stable, their spectrum lies in $\{z : \operatorname{Re} z < 0\}$. Verifying this fact is a delicate problem. We have given conditions that ensure that this property is satisfied. These results will be developed subsequently. The initial matrices A_0 and B_0 are the Jacobian matrices of F at x_0 such that $DF(x)$ exists, and at u_0 such that $DF(u)$ exists, respectively, x_0 and u_0 are the initial values of x and u.

Consider system (1). The computational procedure can be summarized as follows.

STEP 1. Compute

$$A_0 = DF(x_0), \qquad B_0 = DF(u_0).$$

STEP 2. Compute A_1 and B_1 from the solution of equation

$$\frac{dy}{dt} = A_0 y(t) + B_0 v(t), \qquad y(0) = x_0, \tag{16}$$

which is

$$y(t) = e^{tA_0} x_0 + \int_0^t e^{(t-s)A_0} B_0 v(s)\, ds, \tag{17}$$

by minimizing the functional

$$G(A,B) = \int_0^{+\infty} \|F(y(t), v(t)) - Ay(t) - Bv(t)\|^2\, dt. \tag{18}$$

A_1 and B_1 are uniquely determined by system (15), where x is replaced by y and $v(t)$ is the excitation at time t.

From this point on, the matrices determined by the procedure are no longer Jacobian matrices of F at a given point. In order to continue, it is necessary that the above conditions be satisfied at each step.

Let us first assume that this holds. Then, the procedure works as follows.

STEP 3. Assuming that $A_1, \dots, A_{j-1}$ and $B_1, \dots, B_{j-1}$ have been computed, to compute A_j from A_{j-1} and B_j from B_{j-1}, we first solve

$$\frac{dy}{dt} = \left[A_{(j-1)}\right] y(t) + \left[B_{(j-1)}\right] v(t), \qquad y(0) = x_0. \tag{19}$$

The solution y_j of this equation is

$$y_j(t) = e^{tA_{(j-1)}} x_0 + \int_0^t e^{(t-s)A_{(j-1)}} B_{(j-1)} v_j(s)\, ds. \tag{20}$$

The minimization of the functional

$$G_j(A,B) = \int_0^{+\infty} \|F(y_j(t), v_j(t)) - Ay_j(t) - Bv_j(t)\|^2\, dt \tag{21}$$

gives

$$[A_j] = \left[\int_0^{+\infty} [F(y_j(t), v_j(t))][y_j(t)]^\top\, dt - (B_j)(\Phi(v_j, y_j))\right] [\Gamma(y_j)]^{-1}, \tag{22}$$

$$[B_j] = \left[\int_0^{+\infty} [F(y_j(t), v_j(t))][v_j(t)]^\top\, dt - (A_j)(\Phi(y_j, v_j))\right] [\Gamma(v_j)]^{-1}. \tag{23}$$

If the sequences (A_j, B_j) converge, then the limit $(\tilde{A}, \tilde{B})$ is by definition the optimal approximation of $F(x(t), u(t))$ at (x_0, u). The optimal matrices depend on x_0.

3. PROPERTIES OF THE PROCEDURE

We will now consider the situations where the procedure converges.

3.1. Case Where the Application F is Linear

If F is linear, then the procedure gives $\tilde{A}$ and $\tilde{B}$ at the first iteration. The optimal approximation of a linear system is the system itself.

3.2. Case Where the System is the Sum of Linear and Nonlinear Terms

STEP 1. Consider the nonlinear system with a nonlinearity of the form

$$F(x(t),u(t)) = Mx(t) + F^*(x(t),u(t)), \tag{24}$$

where M is linear.

Using equations (10) and (14), we obtain for A_1 and B_1,

$$\begin{aligned} A_1\Gamma_A(x) + B_1\Phi_A(u,x) &= \int_0^{+\infty} [Mx(t) + F^*(x(t),u(t))][x(t)]^\top \, dt, \\ B_1\Gamma_B(u) + A_1\Phi_B(x,u) &= \int_0^{+\infty} [Mx(t) + F^*(x(t),u(t))][u(t)]^\top \, dt. \end{aligned} \tag{25}$$

The computation of the matrices A_1 and B_1 gives

$$\begin{aligned} A_1 &= M + [\Psi_A^*(x,u) - B_1\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_1 &= [M\Phi_B(x,u)][\Gamma_B(u)]^{-1} + [\Psi_B^*(x,u) - A_1\Phi_B(x,u)][\Gamma_B(u)]^{-1}, \end{aligned} \tag{26}$$

with

$$\begin{aligned} \Psi_A^*(x,u) &= \int_0^{+\infty} [F^*(x(t),u(t))][x(t)]^\top \, dt, \\ \Psi_B^*(x,u) &= \int_0^{+\infty} [F^*(x(t),u(t))][u(t)]^\top \, dt. \end{aligned} \tag{27}$$

Hence,

$$\begin{aligned} A_1 &= M + A_1^*, \\ B_1 &= [M\Phi_B(x,u)][\Gamma_B(u)]^{-1} + B_1^*, \end{aligned} \tag{28}$$

where

$$\begin{aligned} A_1^* &= [\Psi_A^*(x,u) - B_1\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_1^* &= [\Psi_B^*(x,u) - A_1\Phi_B(x,u)][\Gamma_B(u)]^{-1}. \end{aligned} \tag{29}$$

Then, for all j, we have

$$\begin{aligned} A_j &= M + A_j^*, \\ B_j &= [M\Phi_B(x,u)][\Gamma_B(u)]^{-1} + B_j^*, \end{aligned} \tag{30}$$

where

$$\begin{aligned} A_j^* &= [\Psi_A^*(x,u) - B_j\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_j^* &= [\Psi_B^*(x,u) - A_j\Phi_B(x,u)][\Gamma_B(u)]^{-1}. \end{aligned} \tag{31}$$

STEP 2. Consider the nonlinear system with a nonlinearity of the form

$$F(x(t),u(t)) = F^*(x(t),u(t)) + Lu(t), \tag{32}$$

where L is linear. Using equations (10) and (14), we obtain, for A_1 and B_1,

$$\begin{aligned} A_1\Gamma_A(x) + B_1\Phi_A(u,x) &= \int_0^{+\infty} [F^*(x(t),u(t)) + Lx(t)][x(t)]^\top \, dt, \\ B_1\Gamma_B(u) + A_1\Phi_B(x,u) &= \int_0^{+\infty} [F^*(x(t),u(t)) + Lx(t)][u(t)]^\top \, dt. \end{aligned} \tag{33}$$

The computation of the matrices A_1 and B_1 gives

$$\begin{aligned} A_1 &= [\Phi_A^*(x,u) - B_1\Phi_A(u,x)][\Gamma_A(x)]^{-1} + [L\Phi_a(u,x)][\Gamma_A(x)]^{-1}, \\ B_1 &= [\Psi_B^*(x,u) - A_1\Phi_B(x,u)][\Gamma_B(u)]^{-1} + L, \end{aligned} \tag{34}$$

with

$$\begin{aligned} \Psi_A^*(x,u) &= \int_0^{+\infty} [F^*(x(t),u(t))][x(t)]^\top \, dt, \\ \Psi_B^*(x,u) &= \int_0^{+\infty} [F^*(x(t),u(t))][u(t)]^\top \, dt. \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} A_1 &= A_1^* + [L\Phi_a(u,x)][\Gamma_A(x)]^{-1}, \\ B_1 &= B_1^* + L, \end{aligned} \tag{36}$$

where

$$\begin{aligned} A_1^* &= [\Psi_A^*(x,u) - B_1\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_1^* &= [\Psi_B^*(x,u) - A_1\Phi_B(x,u)][\Gamma_B(u)]^{-1}. \end{aligned} \tag{37}$$

Then, for all j, we have

$$\begin{aligned} A_j &= A_j^* + [L\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_j &= B_j^* + L, \end{aligned} \tag{38}$$

where

$$\begin{aligned} A_j^* &= [\Psi_A^*(x,u) - B_j\Phi_A(u,x)][\Gamma_A(x)]^{-1}, \\ B_j^* &= [\Psi_B^*(x,u) - A_j\Phi_B(x,u)][\Gamma_B(u)]^{-1}. \end{aligned} \tag{39}$$

In particular, if some components of F are linear, then the corresponding components of F^* are zero, and the corresponding components of A_j and B_j are those of F.

4. ORDER OF THE APPROXIMATION

We will now evaluate the functional

$$G(A,B) = \int_0^{+\infty} \|f(y(t),v(t)) - Ay(t) - Bv(t)\|^2 \, dt, \tag{40}$$

where (A,B) are any matrices. Starting from arbitrary matrices (A_0,B_0), the first matrices (A_1,B_1) obtained in the optimal procedure minimize the functional

$$\int_0^{+\infty} \|F(y_0(t),v(t)) - Ay_0(t) - Bv(t)\|^2 \, dt, \tag{41}$$

where $y_0(t)$ is the solution of the system

$$\frac{dy}{dt} = A_0 y(t) + B_0 v(t), \qquad y(0) = x_0. \tag{42}$$

We have the following relationship between (A_1,B_1) and (A,B):

$$\int_0^{+\infty} \|F(y_0(t),v(t)) - A_1 y_0(t) - B_1 v(t)\|^2 \, dt \le \int_0^{+\infty} \|F(y_0(t),v(t)) - Ay_0(t) - Bv(t)\|^2 \, dt. \tag{43}$$

Then, between (A_j,B_j) and (A,B), we have

$$\begin{aligned} &\int_0^{+\infty} \|F(y_j(t),v(t)) - (A_{j+1})y_j(t) - (B_{j+1})v(t)\|^2 \, dt \\ &\qquad \le \int_0^{+\infty} \|F(y_j(t),v(t)) - Ay_j(t) - Bv(t)\|^2 \, dt, \end{aligned} \tag{44}$$

where

$$y_j(t) = e^{tA_{(j-1)}} x_0 + \int_0^t e^{(t-s)A_{(j-1)}} B_{(j-1)} v(s)\, ds \tag{45}$$

is the solution of the system

$$\frac{dy}{dt} = \left[A_{(j-1)}\right] y(t) + \left[B_{(j-1)}\right] v(t), \qquad y(0) = x_0. \tag{46}$$

In the limit ($j \to +\infty$), we obtain

$$\int_0^{+\infty} \left\|F(\tilde{y}(t), v(t)) - \tilde{A}\tilde{y}(t) - \tilde{B}v(t)\right\|^2 dt \le \int_0^{+\infty} \left\|F(\tilde{y}(t), v(t)) - A\tilde{y}(t) - Bv(t)\right\|^2 dt, \tag{47}$$

where

$$\tilde{y}(t) = e^{t\tilde{A}} x_0 + \int_0^t e^{(t-s)\tilde{A}} \tilde{B} v(s)\, ds \tag{48}$$

is the solution of

$$\frac{dy}{dt} = \tilde{A} y(t) + \tilde{B} v(t), \qquad y(0) = x_0. \tag{49}$$

So,

$$\begin{aligned} &\int_0^{+\infty} \left\|F(\tilde{y}(t), v(t)) - \tilde{A}\tilde{y}(t) - \tilde{B}v(t)\right\|^2 dt \\ &= \inf_{\substack{\forall (A,B)\in \mathcal{M}_n(\mathbb{R}) \\ \operatorname{Re}\sigma(A,B)\in]-\infty,0[}} \left[\int_0^{+\infty} \|F(\tilde{y}(t), v(t)) - A\tilde{y}(t) - Bv(t)\|^2\, dt\right]. \end{aligned} \tag{50}$$

Now, consider the case

$$\frac{dx}{dt} = F(x(t)) + \text{constant}, \tag{51}$$

which is possible if $v(t)$ is constant. In this case, the function $v(t)$ can be written

$$v(t) = [E] H(t), \tag{52}$$

where E is a constant, independent of time and $H(t)$ is the Heaviside function

$$H(t) = \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Hence,

$$\int_0^{+\infty} \left\|F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t)\right\|^2 dt \le \int_0^{+\infty} \|F(\tilde{y}(t)) - A\tilde{y}(t)\|^2\, dt, \tag{53}$$

in particular, for $A = DF(0)$, we have

$$\int_0^{+\infty} \left\|F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t)\right\|^2 dt \le \int_0^{+\infty} \|F(\tilde{y}(t)) - DF(0)\tilde{y}(t)\|^2\, dt. \tag{54}$$

With the assumptions $\|\tilde{y}(t)\| \le C\|x_0\|$ and $\|F(x) - DF(0)x\| = O(\|x\|^2)$, we obtain

$$\int_0^{+\infty} \left\|F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t)\right\|^2 dt \le O\left(\|x_0\|^2\right)^2. \tag{55}$$

We will now evaluate the difference $\|x(t) - \tilde{y}(t)\|$ where x is the solution of equation (1) and $\tilde{y}$ that of the optimal linear equation, both having the same initial value. We have

$$\frac{dx}{dt} - \frac{d\tilde{y}}{dt} = F(x(t)) - \tilde{A}\tilde{y}(t) = F(x(t)) - F(\tilde{y}(t)) + F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t). \tag{56}$$

From Assumption (H3) in Section 2.1, we have

$$\frac{d}{dt}\|x(t) - \tilde{y}(t)\| \leq \gamma\|x(t) - \tilde{y}(t)\| + \left\|F(\tilde{y}(t)) - \tilde{A}\tilde{y}(t)\right\|, \tag{57}$$

and using the Gronwall's lemma, we obtain

$$\begin{aligned} \|x(t) - \tilde{y}(t)\| &\leq \int_0^t e^{\gamma t} \left\|F(\tilde{y}(\tau)) - \tilde{A}\tilde{y}(\tau)\right\|^2 d\tau \\ &\leq \left(\int_0^t e^{2\gamma(t-\tau)}\, d\tau\right)^{1/2} \left(\int_0^t \left\|F(\tilde{y}(\tau)) - \tilde{A}\tilde{y}(\tau)\right\|^2 d\tau\right)^{1/2}. \end{aligned} \tag{58}$$

For every $T > 0$, there exists $M \geq 0$ such that

$$\|x(t) - \tilde{y}(t)\| \leq M\left(\|x_0\|^2\right), \qquad \text{for } 0 \leq t \leq T, \tag{59}$$

and every x_0 in the neighborhood of 0, independent of T.

The proposed approximation is of order two or higher with respect to the initial value. More generally, it has the same order as the nonlinearity.

5. APPLICATIONS

We present first the computational procedure of the least-square approximation.

5.1. Computational Procedure

The computational procedure is based on the algorithm presented in Section 2.3.2, and written in Fortran language. The differential equations have been solved using the fourth-order Runge-Kutta method [22].

INPUT. $[x_0, u, A_0, B_0, \varepsilon]$.

LEVEL 1. Computation of A_1 in terms of A_0 and B_1 in terms of B_0

$$\begin{aligned} A_{(1)} = &\left[\int_0^{+\infty} [F(y_1(t), u(t))][y_1(t)]^\top dt - B_{(1)} \left[\int_0^{+\infty} [u(t)][y_1(t)]^\top dt\right]\right] \\ &\times \left[\int_0^{+\infty} [y_1(t)][y_1(t)]^\top dt\right]^{-1}, \end{aligned} \tag{60}$$

$$\begin{aligned} B_{(1)} = &\left[\int_0^{+\infty} [F(y_1(t), u(t))][u(t)]^\top dt - A_{(1)} \left[\int_0^{+\infty} [y_1(t)][u(t)]^\top dt\right]\right] \\ &\times \left[\int_0^{+\infty} [u(t)][u(t)]^\top dt\right]^{-1}, \end{aligned} \tag{61}$$

where

$$y_1(t) = e^{tA_0}x_0 + \int_0^t e^{(t-s)A_0} B_0 u(s)\, ds. \tag{62}$$

LEVEL 2. Computation of $A_{(j)}$ in terms of $A_{(j-1)}$ and $B_{(j)}$ in terms of $B_{(j-1)}$

$$\begin{aligned} A_{(j)} = &\left[\int_0^{+\infty} [F(y_j(t), u(t))][y_j(t)]^\top dt - B_{(j)} \left[\int_0^{+\infty} [u(t)][y_j(t)]^\top dt\right]\right] \\ &\times \left[\int_0^{+\infty} [y_j(t)][y_j(t)]^\top dt\right]^{-1}, \end{aligned} \tag{63}$$

$$\begin{aligned} B_{(j)} = &\left[\int_0^{+\infty} [F(y_j(t), u(t))][u(t)]^\top dt - A_{(j)} \left[\int_0^{+\infty} [y_j(t)][u(t)]^\top dt\right]\right] \\ &\times \left[\int_0^{+\infty} [u(t)][u(t)]^\top dt\right]^{-1}, \end{aligned} \tag{64}$$

where

$$y_j(t) = e^{tA_{(j-1)}}x_0 + \int_0^{+\infty} e^{(t-s)A_{(j-1)}} B_{(j-1)} u(s)\, ds. \tag{65}$$

LEVEL 3. Computation of

$$\left\| A_{(j)} - A_{(j-1)} \right\|, \tag{66}$$
$$\left\| B_{(j)} - B_{(j-1)} \right\|. \tag{67}$$

LEVEL 4. If

$$\left\| A_{(j)} - A_{(j-1)} \right\| < \varepsilon, \tag{68}$$
$$\left\| B_{(j)} - B_{(j-1)} \right\| < \varepsilon, \tag{69}$$

where ε is the desired level of approximation, then set $\tilde{A} = A_{(j)}$ and $\tilde{B} = B_{(j)}$. $(\tilde{A}, \tilde{B})$ constitutes the optimal approximation of F at (x_0, u). Else, set $A_{(j-1)} = A_{(j)}$, $B_{(j-1)} = B_{(j)}$ and go to Level 2.

5.2. Example

Since the main purpose of the following simple application is to illustrate the usefulness of the theory presented in the above sections, we have chosen to study the time evolution of an electrical circuit (Figure 2) containing a diode whose characteristic (i, v) is nonlinear [23].

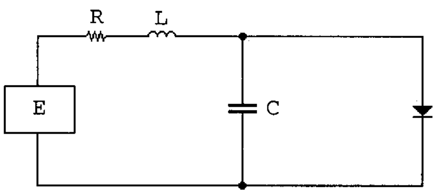

Figure 2. Electronic circuit where a nonlinear diode is the source of excitation.

The model of the diode can be written

$$i = f(v) = \begin{cases} av + bv^2, & v \geq 0, \\ 0, & v < 0, \end{cases}$$

a and b are constants. The state equations can be written

$$\begin{aligned} \frac{dx}{dt} &= \frac{E}{L} - \frac{R}{L}x - \frac{1}{L}y, \\ \frac{dy}{dt} &= \frac{1}{C}x - \frac{a}{C}y - \frac{b}{C}y^2, \qquad (x_0, y_0) = (0,0), \end{aligned} \tag{70}$$

where $x = i_L$ is the self current, $y = v_C$ is the potential drop in the condenser, and $u(t) = E/L$. The parameters of the circuit are

$$\begin{aligned} R &= 100\,\Omega, & a &= 3.5 \times 10^{-3}\,\mathrm{A/V}, \\ C &= 5.10^{-6}\,\mathrm{F}, & b &= 10^{-2}\,\mathrm{A/V^2}, \\ L &= 0.5 \times 10^{-3}\,\mathrm{H}. \end{aligned} \tag{71}$$

Case where the excitation is constant (continuous)

In this case the source of excitation is

$$E = 20\,\mathrm{V}. \tag{72}$$

System (70) is as follows:

$$\begin{aligned} \frac{dx}{dt} &= -2\times 10^5 x - 2\times 10^3 y + u(t), \\ \frac{dy}{dt} &= 2\times 10^5 x - 7\times 10^2 y - 2\times 10^3 y^2, \qquad (x_0, y_0) = (0,0), \end{aligned} \tag{73}$$

where

$$u(t) = \frac{E}{L} H(t) = [40000] H(t), \tag{74}$$

$H(t)$ is the Heaviside function.

The linearization of F at $(x_0, y_0) = (0,0)$ gives

$$DF(x_0, y_0) = \begin{bmatrix} -2\times 10^5 & -2\times 10^3 \\ 2\times 10^5 & -7\times 10^2 \end{bmatrix} \quad \text{and} \quad DF(u_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{75}$$

After ten iterations, the computational procedure gives ($\varepsilon = 10^{-6}$)

$$\tilde{A} = \begin{bmatrix} -2\times 10^5 & -2\times 10^3 \\ 2.5033\times 10^5 & -0.1042\times 10^3 \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{76}$$

and we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2\times 10^5 & -2\times 10^3 \\ 2.5033\times 10^5 & -0.1042\times 10^3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t). \tag{77}$$

REMARK 1. Note that the first equation of the nonlinear system (73) remains without changes after least-square approximation.

Table 1 shows the value of the solution of systems (73) and (77) and the quadratic error. The quadratic error is defined by

$$\mathrm{Er} = \sum_{i=1}^{n} \|y - \tilde{y}\|^2, \tag{78}$$

where y is the solution of the nonlinear system and $\tilde{y}$ the solution of the optimal system.

Figures 3–5 represent, as a function of time, respectively, the excitation $u(t)$, the graphs of the solutions $(x(t), y(t))$ of systems (73) and (77), and the quadratic error between the nonlinear system (73) and the optimal linear system (77).

Table 1. Values of the solution at time $t \in [0, T]$ of the nonlinear system (73) and the optimal linear system (77) and the quadratic error Er.

t	X nl(t)	X lin(t)	Y nl(t)	Y lin(t)	Er
0	0	0	0	0	0
1E−04	1.72422E−01	1.73106E−01	2.84706E+00	2.76327E+00	7.13E−003
2E−04	1.63403E−01	1.64472E−01	3.67573E+00	3.57615E+00	1.01E−002
3E−04	1.61904E−01	1.62118E−01	3.81803E+00	3.79238E+00	6.68E−004
4E−04	1.61676E−01	1.61492E−01	3.84306E+00	3.85015E+00	5.29E−005
5E−04	1.61642E−01	1.61340E−01	3.84688E+00	3.86565E+00	3.63E−004
6E−04	1.61505E−01	1.61307E−01	3.84768E+00	3.86986E+00	5.00E−004
7E−04	1.61552E−01	1.61306E−01	3.84773E+00	3.87099E+00	5.52E−004
8E−04	1.61577E−01	1.61186E−01	3.84772E+00	3.87145E+00	5.82E−004
9E−04	1.61594E−01	1.61205E−01	3.84770E+00	3.87151E+00	5.85E−004
10E−04	1.61439E−01	1.61261E−01	3.84788E+00	3.87146E+00	5.64E−004

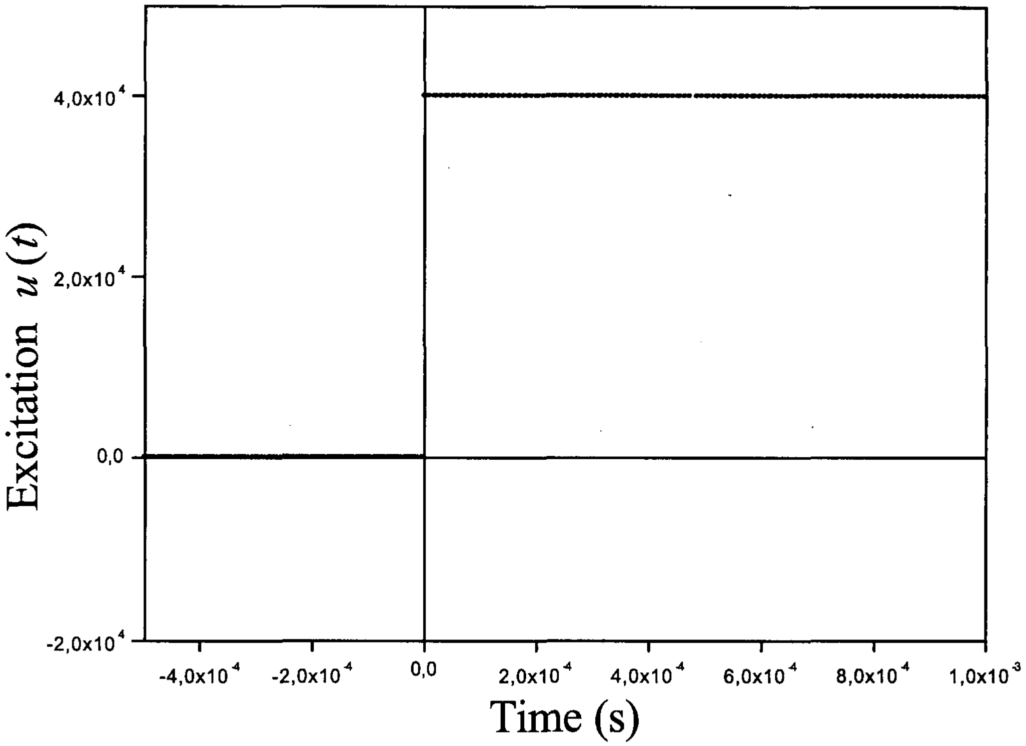

Figure 3. Excitation u as a function of time $u(t) = [40000]H(t)$.

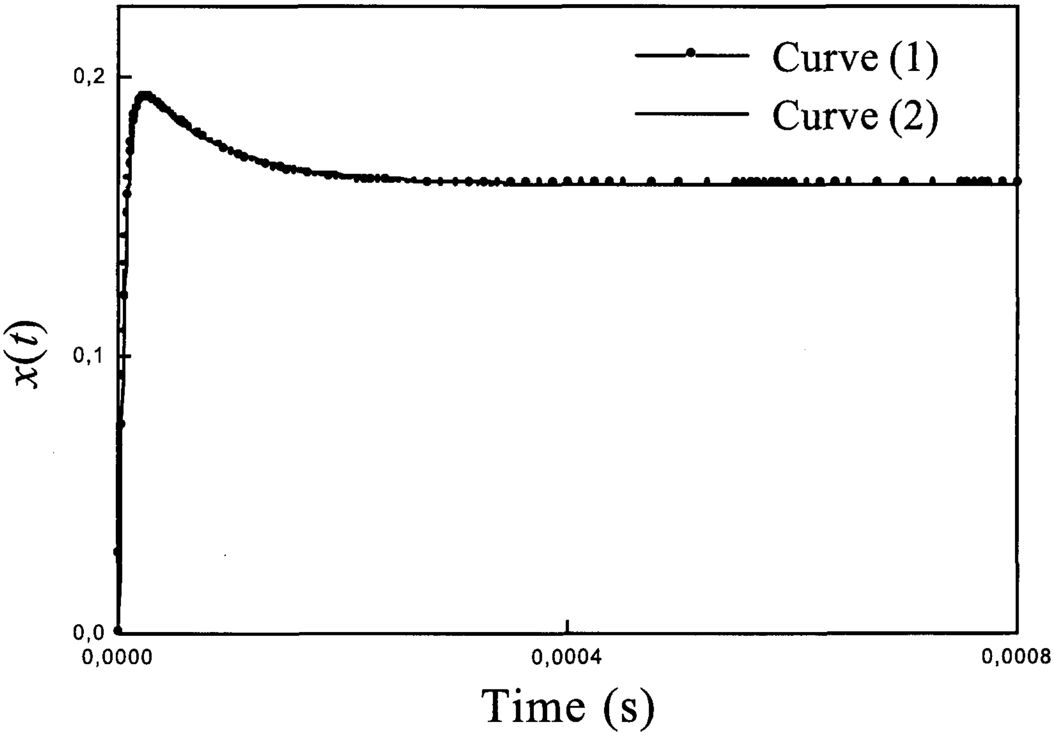

Figure 4. Variation of $x(t)$ and $y(t)$ as a function of time for the initial conditions $(x_0, y_0) = (0, 0)$. Curve (1) corresponds to the solution of system (73). Curve (2) corresponds to the solution of system (77).

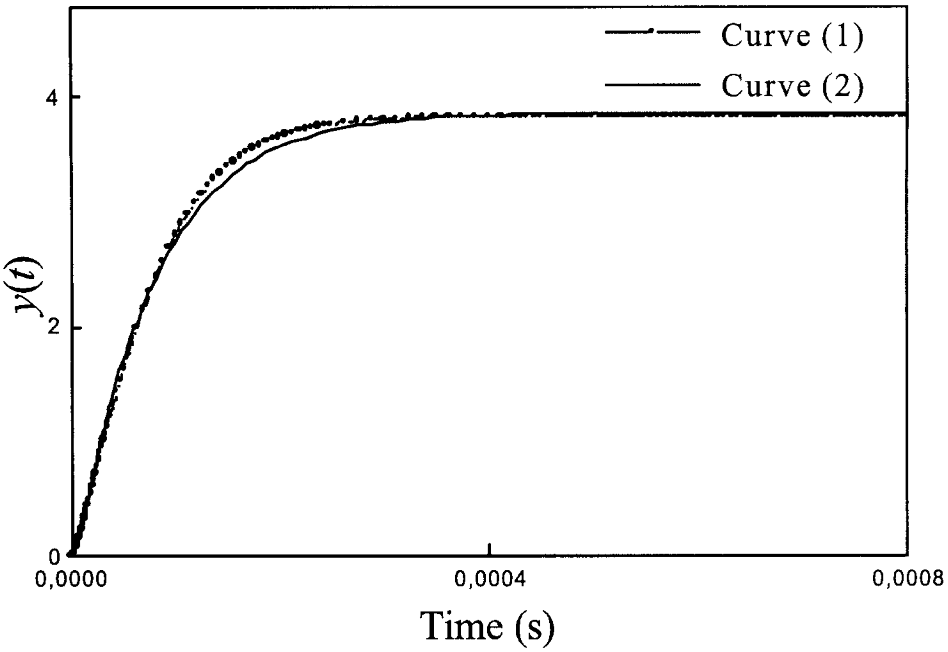

Figure 4. (cont.)

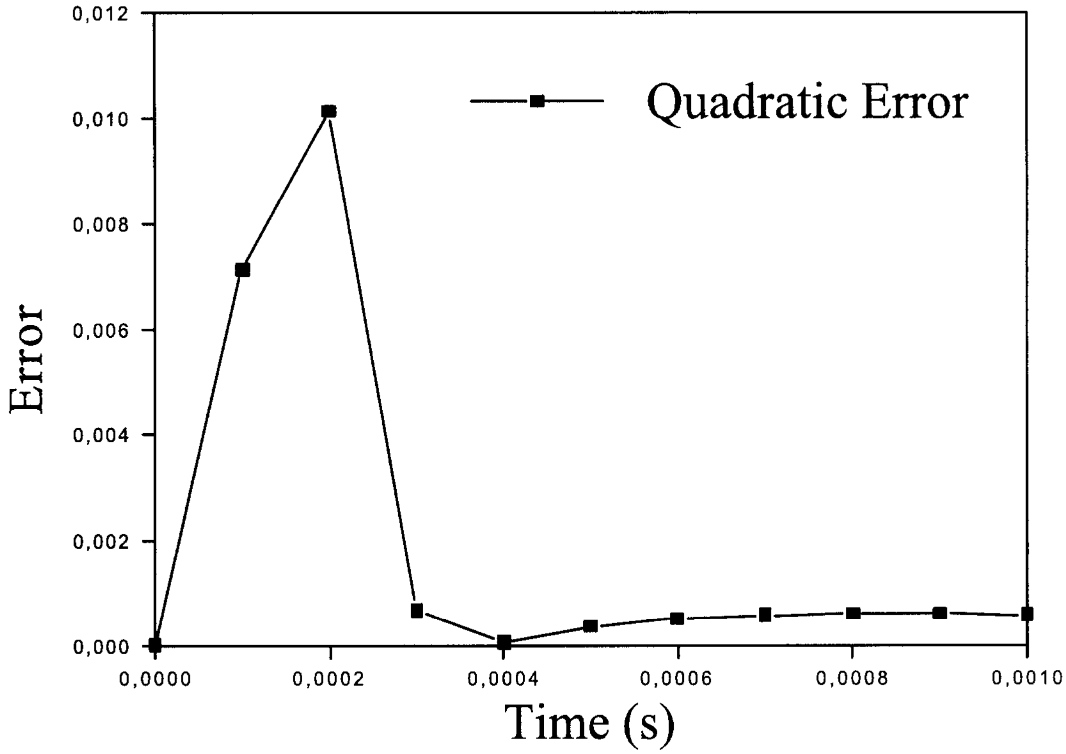

Figure 5. The quadratic error as a function of time between the nonlinear system (73) and the optimal linear system (77).

Case where the excitation is periodic

Now, we consider the case when the source of excitation is

$$E = E_0 \cos(\omega t). \tag{79}$$

$E_0 = 0.2V$ and $\omega = 100\pi$.

System (70) can be written

$$\frac{dx}{dt} = -2 \times 10^5 x - 2 \times 10^3 y + u(t),$$
$$\frac{dy}{dt} = 2 \times 10^5 x - 7 \times 10^2 y - 2 \times 10^3 y^2, \qquad (x_0, y_0) = (0, 0), \tag{80}$$

where

$$u(t) = \frac{E}{L} H(t) = [400 \cos(100\pi t)] H(t), \tag{81}$$

$H(t)$ is the Heaviside function.

The linearization of F at $(x_0, y_0) = (0, 0)$ gives

$$DF(x_0, y_0) = \begin{bmatrix} -2 \times 10^5 & -2 \times 10^3 \\ 2 \times 10^5 & -7 \times 10^2 \end{bmatrix}, \quad \text{and} \quad DF(u_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{82}$$

After nine iterations, the computational procedure gives ($\varepsilon = 10^{-6}$)

$$\tilde{A} = \begin{bmatrix} -1.2419 \times 10^5 & -2.28346 \times 10^3 \\ 1.97255 \times 10^5 & -6.8753 \times 10^2 \end{bmatrix}, \quad \text{and} \quad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{83}$$

and we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1.2419 \times 10^5 & -2.28346 \times 10^3 \\ 1.97255 \times 10^5 & -6.8753 \times 10^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t). \tag{84}$$

Table 2 shows the solutions of systems (80) and (84) and the quadratic error (78). Figures 6–8 represent, as a function of time, respectively, the excitation $u(t)$, the graphs of the solutions $(x(t), y(t))$ of systems (80) and (84), and the quadratic error between the nonlinear system (80) and the optimal linear system (84).

Table 2. Values of the solution at time $t \in [0, T]$ of the nonlinear system (80) and the optimal linear system (84) and the quadratic error Er.

t	X nl(t)	X lin(t)	Y nl(t)	Y lin(t)	Er
0	0	0	0	0	0
0.5E−02	−1.59972E−04	−1.96094E−04	1.62536E−02	1.07934E−02	3.02E−005
1.0E−02	−3.49912E−04	−5.27810E−04	−1.65070E−01	−1.46489E−01	3.52E−004
1.5E−02	1.81407E−04	1.95020E−04	−1.85078E−02	−1.09764E−02	5.69E−005
2.0E−02	6.65381E−04	5.28538E−04	1.33515E−01	1.46466E−01	1.71E−004
2.5E−02	−1.59350E−04	−1.94360E−04	1.66851E−02	1.12173E−02	3.02E−005
3.0E−02	−3.50826E−04	−5.28388E−04	−1.64964E−01	−1.46466E−01	3.48E−004
3.5E−02	1.80568E−04	1.92727E−04	−1.88741E−02	−1.15339E−02	5.40E−005
4.0E−02	6.65474E−04	5.28696E−04	1.33475E−01	1.46447E−01	1.71E−004
4.5E−02	−1.56746E−04	−1.93170E−04	1.69615E−02	1.14857E−02	3.03E−005
5.0E−02	−3.51546E−04	−5.29226E−04	−1.64867E−01	−1.46413E−01	3.47E−004
5.5E−02	1.78455E−04	1.91591E−04	−1.94425E−02	−1.19050E−02	5.70E−005
6.0E−02	6.65439E−04	5.29190E−04	1.33431E−01	1.46408E−01	1.72E−004
6.5E−02	−1.54873E−04	−1.91240E−04	1.74611E−02	1.19521E−02	3.07E−005
7.0E−02	−3.51945E−04	−5.29709E−04	−1.64776E−01	−1.46389E−01	3.44E−004
7.5E−02	1.76186E−04	1.89558E−04	−2.00784E−02	−1.22870E−02	6.09E−005
8.0E−02	6.65938E−04	5.30249E−04	1.33386E−01	1.46356E−01	1.71E−004
8.5E−02	−1.53263E−04	−1.89243E−04	1.78807E−02	1.25737E−02	2.85E−005
9.0E−02	−3.53212E−04	−5.29514E−04	−1.64710E−01	−1.46351E−01	3.43E−004
9.5E−02	1.74807E−04	1.89259E−04	−2.05478E−02	−1.28338E−02	5.97E−005
1.0E−01	6.66372E−04	5.30640E−04	1.33337E−01	1.46310E−01	1.71E−004

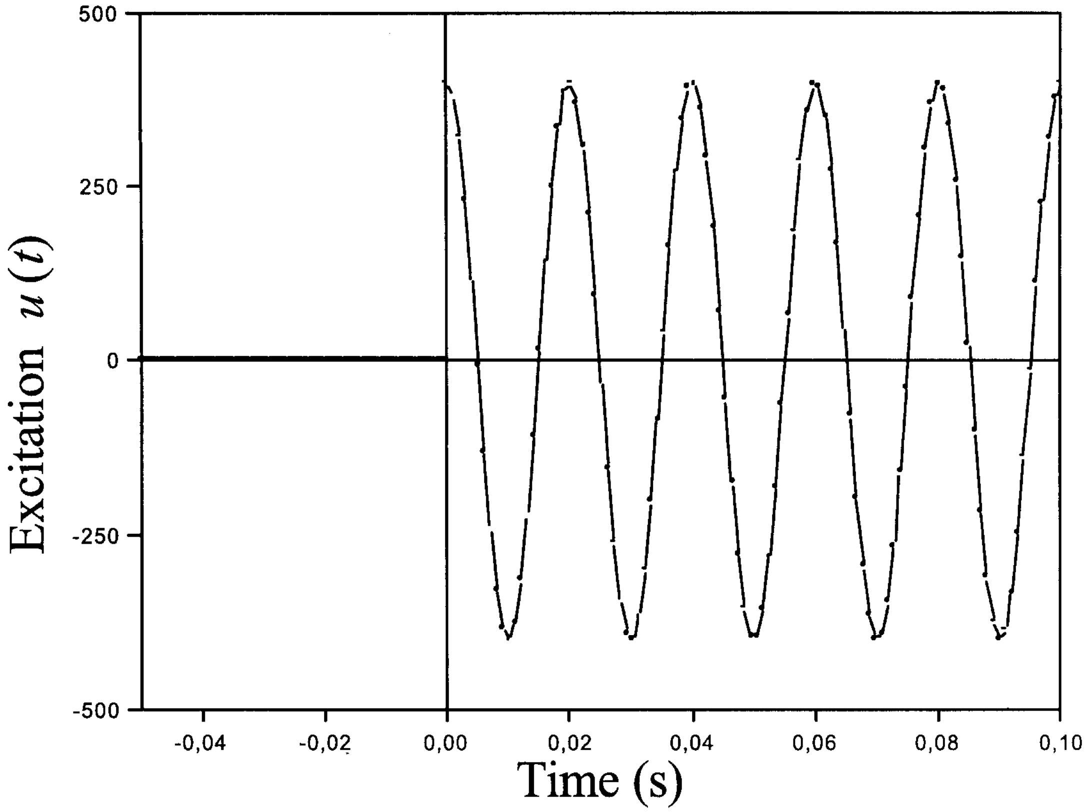

Figure 6. Excitation u as a function of time $u(t) = [400\cos(100\pi t)]H(t)$.

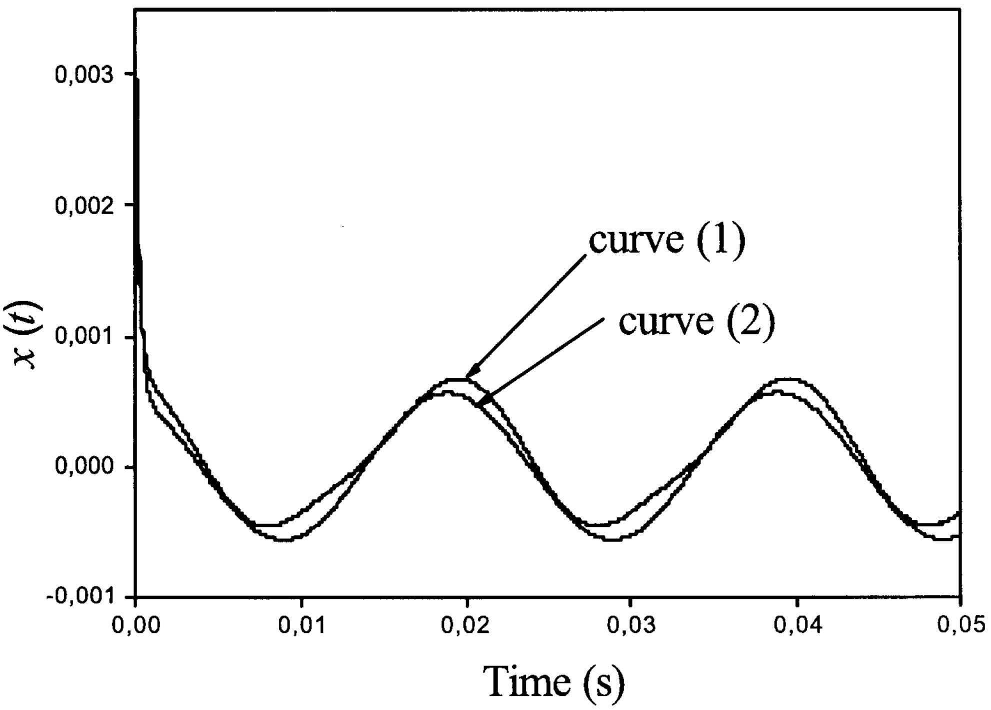

Figure 7. The variation of $x(t)$ and $y(t)$ as a function of time for the initial conditions $(x_0, y_0) = (0, 0)$. Curve (1) corresponds to the solution of system (80). Curve (2) corresponds to the solution of system (84).

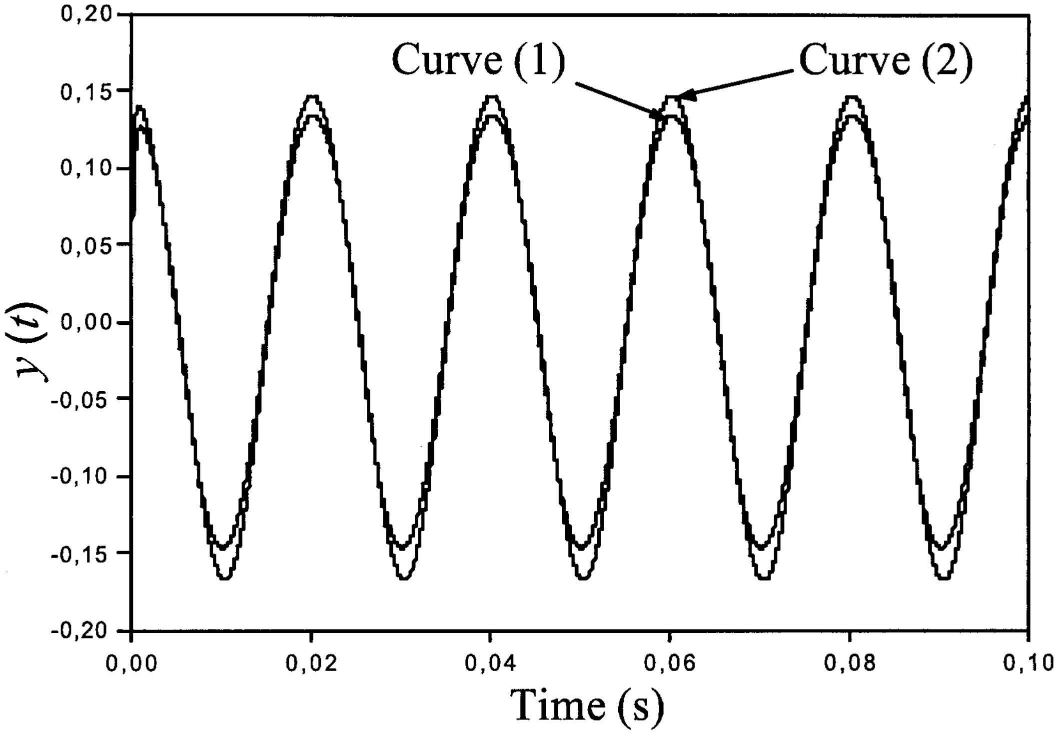

Figure 7. (cont.)

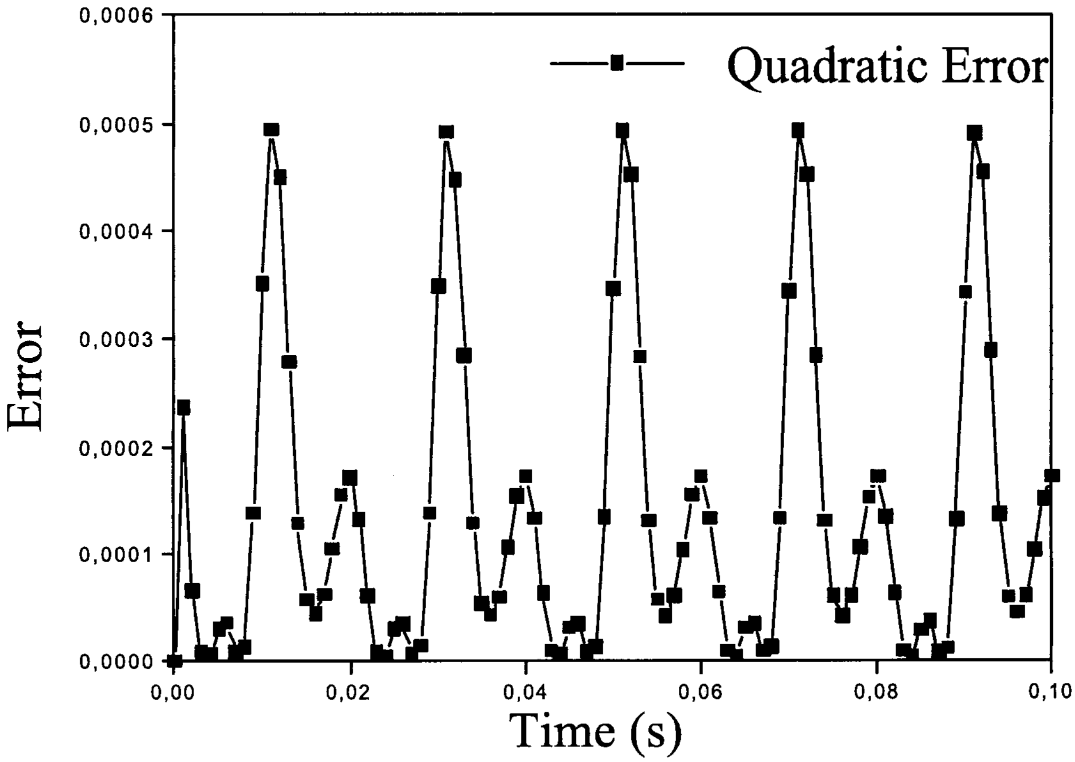

Figure 8. The quadratic error as a function of time between the nonlinear system (80) and the optimal linear system (84).

6. CONCLUSION

As a continuation of earlier papers [3], we have presented in this work further developments regarding the extension of the optimal linearization. The emphasis here was put on the use of the method as an approximation procedure of a nonlinear O.D.E. with excitation.

Our main results stipulate that the approximation is of order two with respect to the initial value, and is generally of the same order of the nonlinearity.

The example presented satisfactory adequacy of approximate results compared to the exact ones. This is confirmed by the computation of the quadratic error which never exceeds 1.1% in the first case. In the second case, the quadratic error order is 0.05%.

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