Detection of the Existence of Bifurcation Surfaces using the Optimal Derivative

S. M. A. Bekkouche¹, T. Benouaz¹ and Martin Bohner²

¹Université Abou Bakr Belkaïd
Laboratoire de Modélisation
B. P. 119, Tlemcen R. P., 13000, Algeria
t_benouaz@mail.univ-tlemcen.dz

²Missouri S&T, Department of Mathematics and Statistics
Rolla, Missouri 65409-0020, USA
bohner@mst.edu

ABSTRACT

The purpose of this paper is to use the optimal derivative method in order to numerically detect the existence of a bifurcation surface of a ratio-dependent model proposed by Arditi and Ginzburg. Modelling by the optimal derivative may reveal some emerging behaviors. Introduced by Arino and Benouaz, it is based on the principle of least square approximation. It is designed as an alternative to the derivative in the sense of Fréchet, essential to the case of equations involving nonregular and in general nondifferentiable functions. We see in particular how this model may contribute to follow numerically the emergence of complex behavior based on very general phenomena such as the change of stability between solutions and other changes in behavior, e.g., the appearance of periodic solutions. Using this model can reveal some original dynamic properties that have not been observed in simple two-dimensional predator-prey models.

Keywords: predator-prey, ratio-dependent system, optimal derivative, stability, bifurcation.

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1 Introduction

The incredible complexities of the operating system given by Arditi and Ginzburg can not be grasped by the simple acquisition of all the parameters that characterize it, as has long been the case when the ecology is confined to a descriptive approach. Beyond a certain point, the presence of many parameters or additional parameters increases the complexity of the model and accumulates uncertainties. It is even possible that this increased uncertainty then makes the model radically diverge from the real system in operation. It is often advisable to prefer a small number of parameters whose uncertainties can counterbalance the quality of the model; in this case, we talk about aggregation. It goes without saying, moreover, that increasing the complexity of the model adds difficulties to its numerical implementation. It
is true that linearization is undoubtedly the main mathematical tool to describe and identify stability properties.

2 Optimal Derivative Review

Consider a nonlinear ordinary differential problem of the form

$$\dot{x} = F(x), \quad x(0) = x_0,$$

where

- $x = (x_1, \ldots, x_n)$ is the unknown function,
- $F = (f_1, \ldots, f_n)$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$,

with the assumptions

(H1) $F(0) = 0$,

(H2) the spectrum $\sigma(DF(x))$ is contained in the set $\{z : \text{Re}z < 0\}$ for every $x \neq 0$, in a neighborhood of 0 for which $DF(x)$ exists,

(H3) $F$ is $\gamma$-Lipschitz continuous.

Given $x_0 \in \mathbb{R}^n$, we choose a first linear map $A_0$. For example, if $F$ is differentiable at $x_0$, then we can take $A_0 = DF(x_0)$ or the derivative value at a point in the vicinity of $x_0$. This is always possible if $F$ is locally Lipschitz. Now, let $y_0$ be the solution of the initial value problem

$$\dot{y} = A_0 y, \quad y(0) = x_0. \quad (2.1)$$

Next, we minimize the functional

$$G(A) = \int_0^\infty \|F(y_0(t)) - Ay_0(t)\|^2 \, dt. \quad (2.2)$$

This minimization problem is always uniquely solvable, and as the optimal linear map minimizing (2.2) we obtain

$$A_1 = \left(\int_0^\infty [F(y_0(t))][y_0(t)]^T \, dt\right) \left(\int_0^\infty [y_0(t)][y_0(t)]^T \, dt\right)^{-1}.$$

Now we define $y_1$ to be the solution of (2.1) with $A_0$ replaced by $A_1$ and we minimize (2.2) with $y_0$ replaced by $y_1$. Then we continue in this way. The optimal derivative $\tilde{A}$ is the limit of the sequence built as such (for details see (Benouaz and Arino, 1996; Benouaz and Arino, 1998; Benouaz, 2000; Benouaz and Arino, 1995c; Benouaz and Arino, 1995a; Benouaz and Arino, 1995b), see also (Benouaz and Bohner, 2007; Benouaz, Bohner and Chikhaoui, 2009; Benouaz, Lassouani, Bekkouche and Bohner, 2011)).
3 Predator-Prey Model of Arditi and Ginzburg

The general ratio-dependent predator-prey model by Arditi and Ginzburg (1989) (see also (Akçakaya, Arditi and Ginzburg, 1995; Berezovskaya, Karev and Arditi, 2001; Jost, Arino and Arditi, 1999)) is given as

\[
\begin{cases}
\dot{N} = Nf(N) - g \left( \frac{N}{P} \right) P \\
\dot{P} = eg \left( \frac{N}{P} \right) P - qP.
\end{cases}
\]

(3.1)

A specific model assumes the familiar form for the logistic growth and the Monod/Holling hyperbolic form for the functional response as

\[ f(x) = r \left( 1 - \frac{x}{K} \right) \quad \text{and} \quad g(x) = \frac{\alpha x}{1 + ahx} \]

and thus

\[ g \left( \frac{N}{P} \right) = \frac{\alpha N}{P + ahN}. \]

With an appropriate change of variables

\[ N = Kx, \quad P = KaHy, \quad t = \frac{\tau}{r} \]

and specific combinations of all positive parameters of the system

\[ \nu = \frac{\alpha}{r}, \quad \mu = \frac{e}{rh}, \quad \gamma = \frac{q}{r}, \]

system (3.1) turns into the system of differential equations

\[
\begin{cases}
\dot{x} = x(1 - x) - \frac{\nu xy}{x + y} \\
\dot{y} = -\gamma y + \frac{\mu xy}{x + y}.
\end{cases}
\]

(3.2)

The parameters \( \gamma, \mu, \) and \( \nu \) are positive, and they have the following meanings: \( \gamma \) is the mortality rate of predators, \( \mu \) is the growth rate of predators, and \( \nu \) is the prey consumption rate.

4 Equilibrium Points

The equilibrium solutions of the system of dynamic equations (3.2) can be written as

\[
\begin{cases}
x = 0 \quad \text{or} \quad y = \frac{x(1 - x)}{x + v - 1} \\
y = 0 \quad \text{or} \quad x = \frac{\gamma y}{\mu - \gamma},
\end{cases}
\]

which implies for \( xy \neq 0 \)

\[ x = \frac{\gamma x(1 - x)}{(\mu - \gamma)(x + v - 1)}, \quad \text{i.e.,} \quad (\mu - \gamma)(x + v - 1) = (1 - x)\gamma, \]
hence
\[ x^* = 1 - \frac{v}{\mu} (\mu - \gamma), \]
and therefore
\[ y^* = \frac{\mu - \gamma}{\gamma} x^* = (\mu - v(\mu - \gamma)) \frac{\mu - \gamma}{\mu \gamma}. \]

In summary,
\[
\begin{cases}
  \text{if } x = 0, \text{ then } y = 0, \text{ then } F \text{ is not defined} \\
  \text{if } y = 0, \text{ then } \\
  \quad \begin{cases} x = 0, \text{ then } F \text{ is not defined} \\
  \quad \text{or} \\
  \quad \text{if } x = 1, \text{ then } A = (1, 0) \text{ is an equilibrium point} \\
  \text{if } xy \neq 0, \text{ then } B = (x^*, y^*) \text{ is an equilibrium point.}
\end{cases}
\end{cases}
\]

**Theorem 4.1.** System (3.2) has three equilibrium points given by
\[ O = (0, 0), \quad A = (1, 0), \quad \text{and } B = \left( \frac{\mu - v(\mu - \gamma)}{\mu}, \frac{\mu(\mu - \gamma) - v(\mu - \gamma)^2}{\mu \gamma} \right). \]

5 Idea of the Problem

Arditi and Ginzburg have introduced the concept of ratio-dependent systems, and it may be considered as an abstraction of the dynamics of the prey and predator populations. This evolution may be better understood by analyzing the behavior of the system over time and its dynamic characteristics. The idea of the problem is to subdivide the parameter space into eight different regions (see Figure 5.1), while picking an arbitrary but fixed mortality rate of predators. The bifurcation boundaries are defined by the following formulas:

- **BA:** \( \mu = v \)
- **BO:** \( v = \frac{\mu}{\mu - \gamma} \)
- **MM:** \[
\begin{cases}
  \mu = \gamma + 1 \\
  v < \gamma + 1,
\end{cases}
\]
- **NN:** \[
\begin{cases}
  v = \gamma + 1 \\
  \mu < \gamma + 1,
\end{cases}
\]
- **H:** \[
\begin{cases}
  v = \frac{\mu}{\mu + \gamma} & (\gamma < \mu < \gamma + 1) \\
  v = \frac{\mu}{\mu - \gamma} & (\mu < \gamma + 1)
\end{cases}
\]
- **L:** \[
\begin{cases}
  v = \frac{0.65 \mu}{\mu - 0.2} + 0.42 \\
  \gamma < \mu < \gamma + 1.
\end{cases}
\]

For a more detailed description of these boundaries, we refer to (Berezovskaya et al., 2001, Theorem A). We also note that the equation for \( v \) in the definition of L was obtained by approximation. We are mainly interested in studying the stability at the border since H is a limit of stability (Andronov–Hopf supercritical bifurcation). Figures 5.1 and 5.2 provide diagrams and bifurcation surfaces.

6 Numerical Simulation

Linear models are used to describe the behavior of systems near the equilibrium. The extensive use of linear systems has been justified by computational reasons: For example the
Figure 5.1: Bifurcation diagram for an arbitrarily fixed $\gamma$.

Figure 5.2: Surfaces of bifurcation BO, L, H and NN (left) and boundaries MM and BA (right).
The linear ODE is the only ODE, with some rare exceptions, which can be solved analytically. The Jacobian matrix of system (3.2) is given by

\[
DF(x, y) = \begin{bmatrix}
1 - 2x - \frac{vy^2}{(x + y)^2} & -\frac{vx^2}{(x + y)^2} \\
\frac{\mu y^2}{(x + y)^2} & -\gamma + \frac{\mu x^2}{(x + y)^2}
\end{bmatrix}.
\]

Therefore, the linear system can be written as

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = DF(B) \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}.
\]

### 6.1 Domain 4

Using the values \((\gamma, \mu, v) = (1.5, 2, 2.7)\), we find \(B = (0.325, 0.1083)\), and the classical linearization is given by the matrix

\[
DF(B) = \begin{bmatrix}
0.18125 & -1.51875 \\
0.1250 & -0.3750
\end{bmatrix}
\]

with eigenvalues

\[
\lambda_1 = -0.0969 + 0.3354i \quad \text{and} \quad \lambda_2 = -0.0969 - 0.3354i.
\]

Using the initial conditions \((x_0, y_0) = (0.3, 0.3)\), the system linearized by the optimal derivative is described by the matrix

\[
\tilde{A} = \begin{bmatrix}
0.3707 & -0.6313 \\
0.5213 & -0.6523
\end{bmatrix}
\]

with eigenvalues

\[
\lambda_1 = -0.1408 + 0.2597i \quad \text{and} \quad \lambda_2 = -0.1408 - 0.2597i.
\]

The eigenvalues are complex conjugates whose real parts are strictly negative. The system is asymptotically stable and shows the equilibrium \(B\) as a stable focus. Figure 6.1 gives the plot of integral lines of the vector fields for the three associated systems while Figure 6.2 shows the time evolution of these three corresponding systems.

### 6.2 Border H

The values chosen in this example are \((\gamma, \mu, v) = (1.5, 2, 22/7)\), which allows to find the equilibrium point \(B = (0.21429, 0.07143)\). The linearization of system (3.2) by the Fréchet derivative gives a matrix that has two imaginary eigenvalues. Then the theorem of stability of linearized systems cannot be used in this case. We have an Andronov–Hopf bifurcation around the equilibrium point. We find

\[
DF(B) = \begin{bmatrix}
0.3750 & -1.7679 \\
0.1250 & -0.3750
\end{bmatrix}
\]

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Figure 6.1: Vector fields of the nonlinear system (3.2) (left), the system (6.1) linearized by the Jacobi method (middle), and the system (6.2) linearized by the optimal derivative method (right).

Figure 6.2: Movement space for the solutions of systems (3.2), (6.1), and (6.2).
with eigenvalues

\[ \lambda_1 = 0.2835i \quad \text{and} \quad \lambda_2 = -0.2835i. \]

The procedure of the optimal derivative after 11 iterations with \((x_0, y_0) = (0.25, 0.06)\) and \(\varepsilon = 10^{-6}\) gives

\[ \tilde{A} = \begin{bmatrix} 0.3543 & -2.5292 \\ 0.0937 & -0.3741 \end{bmatrix} \]  

(6.4)

with eigenvalues

\[ \lambda_1 = -0.0099 + 0.323i \quad \text{and} \quad \lambda_2 = -0.0099 - 0.323i. \]

Figure 6.3: Vector fields of the nonlinear system (3.2) (left), the system (6.3) linearized by the Jacobi method (middle), and the system (6.4) linearized by the optimal derivative method (right).

Figure 6.4: Movement space for the solutions of systems (3.2), (6.3), and (6.4).
6.3 Domain 5

Taking \((\gamma, \mu, \nu) = (1.5, 2.1, 3.143)\), we see that system (3.2) admits \(B = (0.102, 0.0408)\) as an equilibrium point. The linearization method gives the Jacobi matrix

\[
DF(B) = \begin{bmatrix}
0.53943 & -1.60357 \\
0.17143 & -0.42857
\end{bmatrix}
\]

(6.5)

with eigenvalues

\[
\lambda_1 = 0.05543 + 0.2016i \quad \text{and} \quad \lambda_2 = 0.05543 - 0.2016i.
\]

By applying the optimal derivative method with the initial conditions \((x_0, y_0) = (0.102, 0.0407)\), we find

\[
\tilde{A} = \begin{bmatrix}
0.5223 & -1.5138 \\
0.1851 & -0.4333
\end{bmatrix}
\]

(6.6)

with eigenvalues

\[
\lambda_1 = 0.0445 + 0.22784i \quad \text{and} \quad \lambda_2 = 0.0445 - 0.22784i.
\]

The system is unstable and shows the equilibrium point \(B\) as an unstable focus.

Figure 6.5: Vector fields of the nonlinear system (3.2) (left), the system (6.5) linearized by the Jacobi method (middle), and the system (6.6) linearized by the optimal derivative method (right).

7 Conclusion

We studied the qualitative properties of a model proposed by Arditi and Ginzburg (1989) which is commonly used to describe the dynamics of biological systems in which predator and prey interact. The analysis shows quite complicated behavior, in particular the exchange of stability between solutions and the bifurcation of the solution for some critical values called bifurcation points. The qualitative analysis based on the representation of trajectories in space motion shows that they converge to the asymptotically stable equilibrium point \(B\) in the area 4. The boundary \(H\) characterized by critical values is defined as the boundary of a region of asymptotic stability. When parameter values exceed the boundary \(H\) from area 4 to area 5, the equilibrium point \(B\) changes the stability. In other words, this method can be used as an essential and powerful numerical tool for the stability analysis of some biological problems that lead to specific behaviors (ratio-dependent systems).
Figure 6.6: Movement space for the solutions of systems (3.2), (6.5), and (6.6).

References


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